

Macroscopic and microscopic structures of the family tree for decomposable critical branching processes

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Abstract

A decomposable strongly critical Galton-Watson branching process with N types of particles labelled $1, 2, \dots, N$ is considered in which a type i parent may produce individuals of types $j \geq i$ only. This model may be viewed as a stochastic model for the sizes of a geographically structured population occupying N islands, the location of a particle being considered as its type. The newborn particles of island $i \leq N - 1$ either stay at the same island or migrate, just after their birth to the islands $i+1, i+2, \dots, N$. Particles of island N do not migrate. We investigate the structure of the family tree for this process, the distributions of the birth moment and the type of the most recent common ancestor of the individuals existing in the population at a distant moment n .

1 Introduction and main results

We consider a Galton-Watson branching process with N types of particles labelled $1, 2, \dots, N$ and denote by

$$\mathbf{Z}(n) = (Z_1(n), \dots, Z_N(n)), \quad \mathbf{Z}(0) = (1, 0, \dots, 0)$$

the population vector at time $n \in \mathbb{Z}_+ = \{0, 1, \dots\}$. Along with $\mathbf{Z}(n)$ we deal with the process

$$\mathbf{Z}(m, n) = (Z_1(m, n), \dots, Z_N(m, n)),$$

where $Z_i(m, n)$ is the number of type i particles existing in $\mathbf{Z}(\cdot)$ at moment $m < n$ and having nonempty number of descendants at moment n . We agree to write $Z_i(n, n) = Z_i(n)$.

The process $\mathbf{Z}(\cdot, n)$ is called a reduced branching process and can be thought of as the family tree relating the individuals alive at time n . An important

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characteristic of the reduced process is the birth moment β_n of the most recent common ancestor (MRCA) of all individuals existing in the population at moment n defined as

$$\beta_n = \max \{m \leq n-1 : Z_1(m, n) + Z_2(m, n) + \dots + Z_N(m, n) = 1\}.$$

The structure of the family tree and the asymptotic distribution of the birth moment of the MRCA for single-type Galton-Watson branching processes have been studied in [5],[6],[10] and [20]. The case of multitype indecomposable critical Markov branching processes was considered in [19]. Family trees for more general models of branching processes were investigated in [3],[7],[11],[12],[13],[15],[16],[18]. However, the reduced processes for decomposable branching processes have not been analyzed yet. We fill this gap in the present paper and study various properties of the family tree for a particular case of the decomposable Galton-Watson branching processes. Namely, we consider the Galton-Watson branching process with N types of particles labelled $1, 2, \dots, N$ in which a type i parent may produce individuals of types $j \geq i$ only. This model may be viewed as a stochastic model for the sizes of a geographically structured population occupying N islands, the location of a particle being considered as its type. The reproduction laws of particles depend on the island on which the particles are located. The newborn particles of island $i \leq N-1$ either stay at the same island or migrate, just after their birth to the islands $i+1, i+2, \dots, N$. Particles of island N do not migrate.

We investigate the structure of the family tree of this process, the distributions of the birth moment β_n and the type ζ_n of the MRCA. It is shown, in particular, that, as $n \rightarrow \infty$ the conditional reduced process

$$\{\mathbf{Z}(n^t \log n, n), 0 \leq t < 1 | \mathbf{Z}(n) \neq \mathbf{0}\}$$

converges in a certain sense to an N -dimensional inhomogeneous branching process $\{\mathbf{R}(t), 0 \leq t < 1\}$ which, for $t \in [0, 2^{-(N-1)})$ consists of a single particle of type 1 only and for $t \in [2^{-(N-i+1)}, 2^{-(N-i)})$, $i = 2, \dots, N$ consists of type i particles only. These particles are born at moment $t = 2^{-(N-i+1)}$ and die at moment $t = 2^{-(N-i)}$ producing at this moment a random number of descendants having type $\min(i+1, N)$. This gives a macroscopic view on the structure of the family tree of the process.

On the other hand, for each $i = 1, 2, \dots, N-1$ the conditional process

$$\left\{ \mathbf{Z}((y + (\log n)^{-1})n^{2^{-(N-i)}}, n), 0 < y < \infty | \mathbf{Z}(n) \neq \mathbf{0} \right\}$$

converges in a certain sense, as $n \rightarrow \infty$ to a continuous-time homogeneous Markov branching process $\{\mathbf{U}_i(y), 0 \leq y < \infty\}$ which is initiated at time $y = 0$ by a random number of type i particles. These type i particles have an exponential life-length distribution. Dying each of them produces either two particles of type i or one particle of type $i+1$ (both options with probability $1/2$). Particles of type $i+1$ in this process are immortal and produce no offspring. This provides a microscopic view on the structure of the family tree.

To present our results in a more formal way we need some notation. Let \mathbf{e}_i be a vector whose i -th component is equal to one while the remaining are zeros. The first moments of the components of $\mathbf{Z}(n)$ will be denoted as

$$m_{ij}(n) = \mathbf{E}[Z_j(n) | \mathbf{Z}(0) = \mathbf{e}_i]$$

with $m_{ij} = m_{ij}(1)$ being the average number of children of type j produced by a particle of type i .

Since $m_{ij} = 0$ if $i > j$, the mean matrix \mathbf{M} of the decomposable Galton-Watson branching process has the form

$$\mathbf{M} = (m_{ij})_{i,j=1}^N = \begin{pmatrix} m_{11} & m_{12} & \dots & \dots & m_{1N} \\ 0 & m_{22} & \dots & \dots & m_{2N} \\ 0 & 0 & m_{33} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & m_{NN} \end{pmatrix}. \quad (1)$$

To go further it is convenient to deal with the probability generating functions for the reproduction laws of particles

$$h_i(s_1, \dots, s_N) = \mathbf{E}[s_1^{\eta_{i1}} \dots s_N^{\eta_{iN}}], \quad i = 1, 2, \dots, N, \quad (2)$$

where η_{ij} represent the numbers of daughters of type j a mother of type i .

We say that **Hypothesis A** is valid if the N -type decomposable process is strongly critical, i.e. (see [9]),

$$m_{ii} = \mathbf{E}[\eta_{ii}] = 1, \quad i = 1, 2, \dots, N, \quad (3)$$

and, in addition,

$$m_{i,i+1} = \mathbf{E}[\eta_{i,i+1}] \in (0, \infty), \quad i = 1, 2, \dots, N-1, \quad (4)$$

and

$$\mathbf{E}[\eta_{ij}\eta_{ik}] < \infty, \quad i = 1, \dots, N; \quad k, j = i, i+1, \dots, N \quad (5)$$

with

$$b_i = \frac{1}{2} \text{Var}[\eta_{ii}] \in (0, \infty), \quad i = 1, 2, \dots, N. \quad (6)$$

Thus, a particle of the process under consideration is able to produce the direct descendants of its own type, of the next in the order type, and (not necessarily, as direct descendants) of all the remaining in the order types, but not any preceding ones.

To simplify the presentation we fix, from now on $N \geq 2$ and use, when it is convenient the notation

$$\gamma_0 = 0, \quad \gamma_i = \gamma_i(N) = 2^{-(N-i)}, \quad i = 1, 2, \dots, N.$$

We also suppose (if otherwise is not stated) that $\mathbf{Z}(0) = \mathbf{e}_1$, i.e., assume that the branching process under consideration is initiated at time zero by a single particle of type 1.

Let $\xi^{(i)}(j), i = 1, 2, \dots, N; j = 1, 2, \dots$ be a tuple of independent identically distributed random variables with probability generating function

$$f(s) = \mathbf{E} \left[s^{\xi^{(i)}(j)} \right] = 1 - \sqrt{1-s}.$$

By means of the tuple we give a detailed construction of an N -type decomposable branching process $\mathbf{R}(t) = (R_1(t), \dots, R_N(t)), 0 \leq t < 1$, where $R_i(t)$ is the number of type i individuals in the population at moment t . It is this process describes the macroscopic structure of the family tree $\{\mathbf{Z}(m, n), 0 \leq m \leq n\}$ as $n \rightarrow \infty$.

Let $\mathbf{R}(t) = \mathbf{e}_1$ for $\gamma_0 \leq t < \gamma_1$ meaning that the branching process $\mathbf{R}(t)$ starts at $t = 0$ by a single individual of type 1 which survives up to (but not at) moment γ_1 without reproduction. If $\gamma_i \leq t < \gamma_{i+1}, i = 1, 2, \dots, N-1$ then

$$R_k(t) = \begin{cases} \sum_{j=1}^{R_i(\gamma_i-0)} \xi^{(i)}(j) & \text{if } k = i+1 \\ 0 & \text{if } k \neq i+1 \end{cases}.$$

Thus, within the interval $\gamma_i \leq t < \gamma_{i+1}$ the population consists of type $i+1$ particles only. These particles were born at moment $\gamma_i - 0$ by particles of type i evolving without reproduction within the interval $\gamma_{i-1} \leq t < \gamma_i$. More precisely, the j -th particle of type i produces at its death moment $\gamma_i - 0$ a random number $\xi^{(i)}(j)$ children of type $i+1$ and no particles of other types.

In what follows we use the symbol \Rightarrow to denote convergence in the space $D_{[a,b)}(\mathbb{Z}_+^N)$ of cadlag functions $\mathbf{x}(t), a \leq t < b$ with values in \mathbb{Z}_+^N endowed with the metric of Skorokhod topology. Besides, we agree to consider $\mathbf{Z}(x, n)$ as $\mathbf{Z}([x], n)$, where $[x]$ is the integer part of x .

For $0 \leq t \leq 1$ put

$$g_n(t) = 1_{\{0 \leq t < \gamma_1\}} + g_n 1_{\{\gamma_1 \leq t \leq 1\}}$$

where g_n is a positive monotone increasing sequence such that

$$\lim_{n \rightarrow \infty} g_n = \infty \text{ and } \lim_{n \rightarrow \infty} n^{-\varepsilon} g_n = 0 \text{ for any } \varepsilon > 0.$$

Theorem 1 *Let Hypothesis A be valid. Then, as $n \rightarrow \infty$*

1) *the finite-dimensional distributions of the process*

$$\{(\mathbf{Z}(n^t g_n(t), n), 0 \leq t < 1) | \mathbf{Z}(n) \neq \mathbf{0}\}$$

converge to the finite-dimensional distributions of $\{\mathbf{R}(t), 0 \leq t < 1\}$;

2) *for any $i = 0, 1, 2, \dots, N-1$*

$$\mathcal{L} \{(\mathbf{Z}(n^t g_n(t), n), \gamma_i \leq t < \gamma_{i+1}) | \mathbf{Z}(n) \neq \mathbf{0}\} \Rightarrow \mathcal{L} \{\mathbf{R}(t), \gamma_i \leq t < \gamma_{i+1}\}.$$

Remark 1. Theorem 1 shows that the passage to limit under the macroscopic time-scaling $n^t g_n(t)$ transforms the reduced process into an inhomogeneous branching process which consists at any given moment of particles of a

single type only. In particular, the phase transition from type i to type $i + 1$ in the prelimiting process happens, roughly speaking, at moment n^{γ_i} . This gives a macroscopic view on the family tree of the reduced process. The microscopic structure of the family tree described by Theorem 2 below clarifies the nature of the revealed phase transition.

Let $c_{ji}, 1 \leq j \leq i \leq N$ be a tuple of positive numbers in which $c_{ii} = b_i^{-1}$ for $i = 1, 2, \dots, N$ and

$$c_{ji} = \sqrt{b_j^{-1} m_{j,j+1} c_{j+1,i}} \text{ for } j \leq i-1, \quad C_i = c_{1i}. \quad (7)$$

It is not difficult to check that

$$c_{iN} = \left(\frac{1}{b_N} \right)^{1/2^{N-i}} \prod_{j=i}^{N-1} \left(\frac{m_{j,j+1}}{b_j} \right)^{1/2^{j-i+1}}. \quad (8)$$

We now define a tuple of continuous time Markov processes

$$\mathbf{U}_i(y) = (U_{i1}(y), \dots, U_{iN}(y)), \quad 0 \leq y < \infty, \quad i = 1, 2, \dots, N-1,$$

$$\mathbf{U}_N(x) = (U_{N1}(x), \dots, U_{NN}(x)), \quad 0 \leq x < 1.$$

First we describe the structure of the processes $\mathbf{U}_i(y), 1 \leq i \leq N-1$. In this case $U_{ij}(y) \equiv 0, 0 \leq y < \infty, j \neq i, i+1$, while the pair

$$(U_{ii}(y), U_{i,i+1}(y)), 0 \leq y < \infty,$$

constitutes a two-type continuous-time homogeneous Markov branching process with particles of types i and $i+1$. This two-type process is initiated at time $y = 0$ by a random number R_i of type i particles whose distribution is specified by the probability generating function

$$\mathbf{E} [s_i^{R_i}] = \mathbf{E} [s_i^{U_{ii}(0)}] = 1 - (1 - s_i)^{1/2^{i-1}} \quad (9)$$

(in particular, $U_{11}(0) = 1$ with probability 1). The life-length distribution of type i particles is exponential with parameter $2b_i c_{iN}$. Dying each particle of type i produces either two particles of its own type or one particle of type $i+1$ (each option with probability $1/2$). Particles of type $i+1$ of $\mathbf{U}_i(\cdot)$ are immortal and produce no children.

The structure of the N -dimensional process $\mathbf{U}_N(x), 0 \leq x < 1$ is different. If $j < N$ then $U_{Nj}(x) \equiv 0, 0 \leq x < 1$, while the component $U_{NN}(\cdot)$ is a single-type inhomogeneous Markov branching process initiated at time $x = 0$ by a random number R_N of type N individuals distributed in accordance with probability generating function

$$\mathbf{E} [s_N^{R_N}] = \mathbf{E} [s_N^{U_{NN}(0)}] = 1 - (1 - s_N)^{1/2^{N-1}}. \quad (10)$$

The life-length of each of R_N type N initial particles is uniformly distributed on the interval $[0, 1]$. Dying such a particle produces exactly two children of type N and nothing else. If the death moment of the parent particle is x then the life length of each of its offspring has the uniform distribution on the interval $[x, 1]$ (independently of the behavior of other particles and the prehistory of the process). Dying each particle of the process produces exactly two individuals of type N and so on...

We are now ready to formulate one more important result of the paper, describing the microscopic structure of the family tree.

Let l_n be a monotone decreasing sequence such that

$$\lim_{n \rightarrow \infty} l_n = 0 \text{ and } \lim_{n \rightarrow \infty} n^\varepsilon l_n = \infty \text{ for any } \varepsilon > 0.$$

Theorem 2 *Let Hypothesis A be valid. Then, as $n \rightarrow \infty$*

1) *for each $i = 1, 2, \dots, N-1$*

$$\mathcal{L} \{ (\mathbf{Z}((y + l_n)n^{\gamma_i}, n), 0 \leq y < \infty) \mid \mathbf{Z}(n) \neq \mathbf{0} \} \implies \mathcal{L}_{R_i} \{ \mathbf{U}_i(y), 0 \leq y < \infty \},$$

where \mathcal{L}_{R_i} means that $\mathbf{U}_i(\cdot)$ is initiated at time $y = 0$ by a random number R_i particles of type i (with $R_1 \equiv 1$);

$$2) \mathcal{L} \{ (\mathbf{Z}((x + l_n)n, n), 0 \leq x < 1) \mid \mathbf{Z}(n) \neq \mathbf{0} \} \implies \mathcal{L}_{R_N} \{ \mathbf{U}_N(x), 0 \leq x < 1 \},$$

where \mathcal{L}_{R_N} means that $\mathbf{U}_N(\cdot)$ is initiated at time $x = 0$ by a random number R_N particles of type N .

Remark 2. Theorems 1 and 2 reveal an interesting phenomenon in the development of the critical decomposable branching processes which may be expressed in terms of the "island" interpretation of the processes as follows: If the population survives up to a distant moment n , then all surviving individuals are located at this moment on island N and, moreover, at each moment in the past their ancestors were (asymptotically) located not more than on two specific islands.

Basing on the conclusions of Theorems 1 and 2 we give in the next theorem an answer to the following important question: what is the asymptotic distribution of the birth moment of the MRCA for the population survived up to a distant moment n ?

Theorem 3 *Let Hypothesis A be valid. Then*

1)

$$\lim_{n \rightarrow \infty} \mathbf{P}(\beta_n \ll n^{\gamma_1} \mid \mathbf{Z}(n) \neq \mathbf{0}) = 0;$$

2) *if $y \in (0, \infty)$ then for $i = 1, 2, \dots, N-1$*

$$\lim_{n \rightarrow \infty} \mathbf{P}(\beta_n \leq yn^{\gamma_i} \mid \mathbf{Z}(n) \neq \mathbf{0}) = 1 - \frac{1}{2^i} - \frac{1}{2^i} e^{-2b_i c_{iN} y};$$

3) *for $i = 1, 2, \dots, N-1$*

$$\lim_{n \rightarrow \infty} \mathbf{P}(\beta_n \ll n^{\gamma_i} \mid \mathbf{Z}(n) \neq \mathbf{0}) = 1 - \frac{1}{2^{i-1}}; \quad (11)$$

3a) for $i = 1, 2, \dots, N - 1$

$$\lim_{n \rightarrow \infty} \mathbf{P}(n^{\gamma_i} \ll \beta_n \ll n^{\gamma_{i+1}} | \mathbf{Z}(n) \neq \mathbf{0}) = 0; \quad (12)$$

4) for any $x \in (0, 1)$

$$\lim_{n \rightarrow \infty} \mathbf{P}(\beta_n \leq xn | \mathbf{Z}(n) \neq \mathbf{0}) = 1 - \frac{1}{2^{N-1}}(1 - x).$$

Remark 3. As we see by (12), there are time-intervals of increasing orders within each of which the probability to find the MRCA of the population survived up to moment $n \rightarrow \infty$ is negligible compared to the probability for the population to survive up to this moment. Moreover, these time-intervals are separated from each other by the time-intervals of increasing orders within each of which the probability to find the MRCA is strictly positive. Such a phenomena has no analogues for the indecomposable Galton-Watson processes.

Along with the distribution of the birth moment of the MRCA, the type ζ_n of the MRCA of the population survived up to moment n is of interest. The distribution of this random variable is described by the following theorem.

Theorem 4 *Let Hypothesis A be valid. Then, for $i = 1, 2, \dots, N$*

$$p_i = \lim_{n \rightarrow \infty} \mathbf{P}(\zeta_n = i | \mathbf{Z}(n) \neq \mathbf{0}) = \frac{1}{2^i}(1 - \delta_{iN}) + \frac{1}{2^{N-1}} \delta_{iN},$$

where δ_{ij} is the Kroneker symbol.

Observe that $p_{N-1} = p_N$.

Remark 4. The authors of paper [9], which contains several results used in the proofs of our Theorems 1-4, considered a more general case of the strongly critical branching processes. Namely, they prove a number of conditional limit theorems for the case when by a suitable labelling the types of the multitype Galton-Watson process can be grouped into $N \geq 2$ partially ordered classes $\mathcal{C}_1 \rightarrow \mathcal{C}_2 \rightarrow \dots \rightarrow \mathcal{C}_N$ possessing the following properties:

- 1) particle types belonging to any given class, say \mathcal{C}_i , constitute an indecomposable critical branching process with $r_i \geq 1$ types;
- 2) each class \mathcal{C}_i contains a type whose representatives are able to produce offspring in the next class in the order with a positive probability;
- 3) particles with types from $\mathcal{C}_i, i \geq 2$, are unable to produce offspring belonging to the classes $\mathcal{C}_1, \dots, \mathcal{C}_{i-1}$.

The methods used in the present paper may be applied to investigate, for instance, the asymptotic distribution of β_n for such processes. Since the needed arguments are too cumbersome and contain no new ideas, we prefer to concentrate on the case when each class \mathcal{C}_i includes a single type only.

The remainder of the paper is organized as follows. Section 2 contains some preliminary results. In particular, we recall the statements from [8] and [9] describing the asymptotic behavior of the survival probability and the distribution of the number of particles in a strongly critical decomposable branching process.

Section 3 gives a detailed description of the limiting processes. In Sections 4 and 5 we check convergence of one-dimensional and finite-dimensional distributions of the prelimiting processes to the limiting ones. Section 6 contains the proofs of Theorems 1 and 2. Finally, Section 7 is devoted to the proofs of Theorems 3 and 4.

2 Auxiliary results

For any vector $\mathbf{s} = (s_1, \dots, s_p)$ (the dimension will usually be clear from the context) and an integer valued vector $\mathbf{k} = (k_1, \dots, k_p)$ define

$$\mathbf{s}^{\mathbf{k}} = s_1^{k_1}, \dots, s_p^{k_p}.$$

Further, let $\mathbf{1} = (1, \dots, 1)$ be a vector of units. It will be sometimes convenient to write $\mathbf{1}^{(i)}$ for the i -dimensional vector with all its components equal to one.

Let

$$H_n^{(i,N)}(\mathbf{s}) = \mathbf{E} \left[\mathbf{s}^{\mathbf{Z}(n)} | \mathbf{Z}(0) = \mathbf{e}_i \right] = \mathbf{E} \left[s_i^{Z_i(n)} \dots s_N^{Z_N(n)} | \mathbf{Z}(0) = \mathbf{e}_i \right]$$

be the probability generating function for $\mathbf{Z}(n)$ given the process is initiated at time zero by a single particle of type $i \in \{1, 2, \dots, N\}$. Clearly (recall (2)), $H_1^{(i,N)}(\mathbf{s}) = h_i(\mathbf{s})$, $i = 1, \dots, N$. Denote

$$Q_n^{(i,N)}(\mathbf{s}) = 1 - H_n^{(i,N)}(\mathbf{s}), \quad Q_n^{(i,N)} = 1 - H_n^{(i,N)}(\mathbf{0}),$$

put

$$\mathbf{H}_n(\mathbf{s}) = (H_n^{(1,N)}(\mathbf{s}), \dots, H_n^{(N,N)}(\mathbf{s})), \quad \mathbf{Q}_n(\mathbf{s}) = (Q_n^{(1,N)}(\mathbf{s}), \dots, Q_n^{(N,N)}(\mathbf{s}))$$

and set

$$b_{jk}(n) = \mathbf{E} [Z_j(n)Z_k(n) - \delta_{jk}Z_j(n) | \mathbf{Z}(0) = \vec{e}_j].$$

The starting point of our arguments is the following theorem being a simplified combination of the respective results from [8] and [9]:

Theorem 5 *Let $\mathbf{Z}(n), n = 0, 1, \dots$ be a strongly critical decomposable multitype branching process satisfying (1), (3), (4), and (5). Then, as $n \rightarrow \infty$*

$$m_{jj}(n) = 1, \quad m_{ij}(n) \sim a_{ij}n^{j-i}, \quad i < j, \quad (13)$$

$$b_{jk}(n) \sim \hat{a}_{jk}n^{k-j+1}, \quad j \leq k, \quad (14)$$

where a_{ij} and \hat{a}_{jk} are positive constants known explicitly (see [9], Theorem 1).

Besides (see [8], Theorem 1), as $n \rightarrow \infty$

$$Q_n^{(i,N)} = 1 - H_n^{(i,N)}(\mathbf{0}) = \mathbf{P}(\mathbf{Z}(n) \neq \mathbf{0} | \mathbf{Z}(0) = \mathbf{e}_i) \sim c_{iN}n^{-1/2^{N-i}}, \quad (15)$$

where the constants c_{iN} are the same as in (8).

In the sequel we prove the following Yaglom-type limit theorem being a compliment to Theorem 5.

Theorem 6 *Under the conditions of Theorem 5, for any $\lambda > 0$*

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[\exp \left\{ -\lambda \frac{Z_N(n)}{b_N n} \right\} \middle| \mathbf{Z}(n) \neq \mathbf{0}; \mathbf{Z}(0) = \mathbf{e}_i \right] = 1 - \left(\frac{\lambda}{1 + \lambda} \right)^{1/2^{N-i}}. \quad (16)$$

Set $d_{ii} = \sqrt{b_i^{-1} m_{i,i+1}}$, $i = 1, 2, \dots, N-1$ and, for $j = 1, 2, \dots, i-1$ let

$$d_{ji} = \sqrt{b_j^{-1} m_{j,j+1} d_{j+1,i}}, \quad D_i = d_{1i}. \quad (17)$$

Observe that (see (7)) for $k = 0, 1, 2, \dots, i-1$

$$d_{i-k,i} = (b_i m_{i,i+1})^{1/2^{k+1}} c_{i-k,i}, \quad D_i = (b_i m_{i,i+1})^{1/2^i} c_{1i} = (b_i m_{i,i+1})^{1/2^i} C_i. \quad (18)$$

Let $\mathbf{Z}(0) = \mathbf{e}_1$ and denote by

$$T_i = \min \{n \geq 1 : Z_1(n) + Z_2(n) + \dots + Z_i(n) = 0\}$$

the extinction moment of the population generated by the particles of the first i in order types. Let $\eta_{rj}(k, l)$ be the number of daughters of type j of the l -th mother of type r belonging to the k -th generation and

$$W_{pij} = \sum_{r=p}^i \sum_{k=0}^{T_i} \sum_{q=1}^{Z_r(k)} \eta_{rj}(k, q)$$

be the total amount of daughters of type $j \geq i+1$ produced by all particles of types $p, p+1, \dots, i$ ever born in the process if the process is initiated at time $n = 0$ by a single particle of type $p \leq i$. Finally, put

$$W_{pi} = \sum_{j=i+1}^N W_{pij} = \sum_{j=i+1}^N \sum_{r=p}^i \sum_{k=0}^{T_i} \sum_{q=1}^{Z_r(k)} \eta_{rj}(k, q).$$

We know by (15) that

$$Q_n^{(1,i)} = \mathbf{P}(T_i > n) \sim c_{1i} n^{-2^{-(i-1)}}. \quad (19)$$

The next lemma describes the tail distributions of $W_{1i,i+1}$ and W_{1i} .

Lemma 7 *Let Hypothesis A be valid. Then, as $\lambda \downarrow 0$*

$$1 - \mathbf{E} [e^{-\lambda W_{1i,i+1}} | \mathbf{Z}(0) = \mathbf{e}_1] \sim d_{1i} \lambda^{1/2^i} = D_i \lambda^{1/2^i} \quad (20)$$

and there exists a constant $F_i > 0$ such that

$$1 - \mathbf{E} [e^{-\lambda W_{1i}} | \mathbf{Z}(0) = \mathbf{e}_1] \sim F_i \lambda^{1/2^i}. \quad (21)$$

Proof. Set

$$W_{pi,i+1}(n) = \sum_{r=p}^i \sum_{k=0}^n \sum_{q=1}^{Z_r(k)} \eta_{rj}(k, q),$$

denote

$$K_{pi,n}(\mathbf{s}; t) = \mathbf{E} \left[s_p^{Z_p(n)} \dots s_i^{Z_i(n)} t^{W_{pi,i+1}(n)} | \mathbf{Z}(0) = \mathbf{e}_p \right], \quad K_{pi,n}(t) = K_{pi,n}(\mathbf{1}^{(i-p+1)}; t)$$

and put

$$K_{pi}(t) = \mathbf{E} \left[t^{W_{pi,i+1}} | \mathbf{Z}(0) = \mathbf{e}_p \right] = \lim_{n \rightarrow \infty} K_{pi,n}(t)$$

(this limit exists since the random variables $W_{pi,i+1}(n)$, $p = 1, 2, \dots, i$ are nondecreasing in n). Clearly, to prove the lemma it is sufficient to show that, as $t \uparrow 1$

$$1 - K_{1i}(t) = 1 - \mathbf{E} \left[t^{W_{1i,i+1}} | \mathbf{Z}(0) = \mathbf{e}_1 \right] \sim d_{1i}(1 - t)^{1/2^i}.$$

Using properties of branching processes it is not difficult to check that

$$K_{pi,n+1}(\mathbf{s}; t) = h_p \left(K_{pi,n}(\mathbf{s}; t), \dots, K_{ii,n}(\mathbf{s}; t), t, \mathbf{1}^{(N-i-1)} \right)$$

implying

$$K_{pi,n+1}(t) = h_p \left(K_{pi,n}(t), \dots, K_{ii,n}(t), t, \mathbf{1}^{(N-i-1)} \right).$$

and

$$K_{pi}(t) = h_p \left(K_{pi}(t), \dots, K_{ii}(t), t, \mathbf{1}^{(N-i-1)} \right).$$

In particular,

$$K_{ii}(t) = h_i \left(K_{ii}(t), t, \mathbf{1}^{(N-i-1)} \right).$$

Since $\mathbf{E}\eta_{ii} = 1$ and $b_i = \frac{1}{2} \text{Var}\eta_{ii} \in (0, \infty)$, it follows that, as $t \uparrow 1$

$$\begin{aligned} 1 - K_{ii}(t) &= 1 - h_i \left(K_{ii}(t), t, \mathbf{1}^{(N-i-1)} \right) \\ &= 1 - K_{ii}(t) - b_i(1 - K_{ii}(t))^2(1 + o(1)) + m_{i,i+1}(1 - t) \end{aligned}$$

or

$$1 - K_{ii}(t) \sim \sqrt{b_i^{-1} m_{i,i+1}(1 - t)}.$$

This, in particular, proves the statement of the lemma for $i = 1$.

Now we use induction and assume that

$$1 - K_{qi}(t) \sim d_{qi}(1 - t)^{1/2^{i-q+1}}, \quad q = p+1, \dots, i.$$

Then

$$\begin{aligned} 1 - K_{pi}(t) &= 1 - h_p \left(K_{pi}(t), \dots, K_{ii}(t), t, \mathbf{1}^{(N-i-1)} \right) \\ &= 1 - K_{pi}(t) - b_p(1 - K_{pi}(t))^2(1 + o(1)) \\ &\quad + (1 + o(1)) \left(m_{p,p+1}(1 - K_{p+1,i}(t)) + \sum_{q=p+2}^i m_{pq}(1 - K_{qi}(t)) \right) \\ &\quad + (1 + o(1))m_{p,i+1}(1 - t) \end{aligned}$$

implying

$$\begin{aligned} 1 - K_{pi}(t) &\sim \sqrt{b_p^{-1} m_{p,p+1} (1 - K_{p+1,i}(t))} \\ &\sim \sqrt{b_p^{-1} m_{p,p+1} d_{p+1,i}} (1-t)^{1/2^{i-p+1}} = d_{pi} (1-t)^{1/2^{i-p+1}} \end{aligned}$$

and proving (20).

To prove (21) it is necessary to use similar arguments. We omit the details. Lemma 7 is proved.

From now on and till the end of this section we suppose that

$$s_k = \exp(-\lambda_k n^{-2^{-(N-k)}}) = \exp(-\lambda_k n^{-\gamma_k}), \lambda_k > 0, k = 1, 2, \dots, N \quad (22)$$

and, keeping in mind this assumption, study in Lemmas 8-11 the asymptotic behavior of the difference $1 - H_m^{(j,N)}(\mathbf{s})$ when $m, n \rightarrow \infty$.

Lemma 8 *If*

$$m \ll n^{2^{-(N-j)}} = n^{\gamma_j} \quad (23)$$

then for $N > j$

$$\lim_{n \rightarrow \infty} n^{\gamma_j} Q_m^{(j,N)}(\mathbf{s}) = \lambda_j.$$

Proof. Clearly, it is sufficient to prove the statement for $j = 1$ only. Let r be a positive integer such that

$$1 - H_r^{(1,1)}(0) \leq 1 - s_1 \leq 1 - H_{r-1}^{(1,1)}(0).$$

Since $1 - s_1 \sim \lambda_1 n^{-\gamma_1}$ and $1 - H_r^{(1,1)}(0) \sim (b_1 r)^{-1}$ as $n, r \rightarrow \infty$, it follows that $r \sim (b_1 \lambda_1)^{-1} n^{\gamma_1}$. By the branching property of probability generating functions we have for $m \ll n^{\gamma_1}$:

$$\begin{aligned} Q_m^{(1,N)}(\mathbf{s}) &\geq 1 - H_m^{(1,1)}(s_1) \geq 1 - H_m^{(1,1)}(H_r^{(1,1)}(0)) \\ &= 1 - H_{m+r}^{(1,1)}(0) \sim b_1^{-1} (m+r)^{-1} \sim \lambda_1 n^{-\gamma_1}. \end{aligned}$$

Besides,

$$\begin{aligned} Q_m^{(1,N)}(\mathbf{s}) &\leq 1 - H_m^{(1,1)}(s_1) + \mathbf{E} \left[\left(1 - s_2^{Z_2(m)} \dots s_N^{Z_N(m)} \right) | \mathbf{Z}(0) = \mathbf{e}_1 \right] \\ &\leq 1 - H_{m+r-1}^{(1,1)}(0) + \sum_{k=2}^N (1 - s_k) \mathbf{E} [Z_k(m) | \mathbf{Z}(0) = \mathbf{e}_1]. \end{aligned}$$

We know by (13) and (22) that, for a positive constant C

$$\sum_{k=2}^N (1 - s_k) \mathbf{E} [Z_k(m) | \mathbf{Z}(0) = \mathbf{e}_1] \leq C \sum_{k=2}^N \lambda_k n^{-\gamma_k} m^{k-1}$$

which, in view of (23) is negligible with respect to

$$C \max_{2 \leq i \leq N} \lambda_i \times \sum_{k=2}^N n^{-\gamma_k} (n^{\gamma_1})^{k-1} = C \max_{2 \leq i \leq N} \lambda_i \times \sum_{k=2}^N n^{(k-1)2^{-(N-1)} - 2^{-(N-k)}}.$$

Since $k2^{-(N-1)} - 2^{-(N-k)} = 2^{-(N-1)}(k - 2^{k-1}) \leq 0$ for $k \geq 2$, we have

$$n^{2^{-(N-1)}} \sum_{k=2}^N n^{(k-1)2^{-(N-1)} - 2^{-(N-k)}} = \sum_{k=2}^N n^{k2^{-(N-1)} - 2^{-(N-k)}} \leq N - 1.$$

Consequently, $Q_m^{(1,N)}(\mathbf{s}) \sim 1 - H_m^{(1,1)}(s_1) \sim \lambda_1 n^{-\gamma_1}$ as $n \rightarrow \infty$.

This proves the lemma.

In order to formulate the next lemma we introduce a tuple of functions $\phi_i = \phi_i(\lambda_1, \lambda_2)$, $i = 1, 2, \dots, N-1$ solving in the domain $\{\lambda_1 \geq 0, \lambda_2 \geq 0\}$ the differential equations

$$\lambda_1 \frac{\partial \phi_i}{\partial \lambda_1} + 2\lambda_2 \frac{\partial \phi_i}{\partial \lambda_2} = -b_i \phi_i^2 + \phi_i + m_{i,i+1} \lambda_2$$

with the initial conditions

$$\phi_i(\mathbf{0}) = 0, \quad \frac{\partial \phi_i(\mathbf{0})}{\partial \lambda_1} = 1, \quad \frac{\partial \phi_i(\mathbf{0})}{\partial \lambda_2} = m_{i,i+1}.$$

One may check that, for any $y > 0$

$$\frac{\phi_i(\lambda_1 y, \lambda_2 y^2)}{y} = \sqrt{\frac{m_{i,i+1} \lambda_2}{b_i} \frac{b_i \lambda_1 + \sqrt{b_i m_{i,i+1} \lambda_2} \tanh(y \sqrt{b_i m_{i,i+1} \lambda_2})}{b_i \lambda_1 \tanh(y \sqrt{b_i m_{i,i+1} \lambda_2}) + \sqrt{b_i m_{i,i+1} \lambda_2}}}. \quad (24)$$

Lemma 9 *Let condition (22) be valid. If $m \sim y n^{\gamma_i}$, $y > 0$ then*

$$\lim_{n \rightarrow \infty} n^{\gamma_i} Q_m^{(i,N)}(\mathbf{s}) = y^{-1} \phi_i(\lambda_i y, \lambda_{i+1} y^2).$$

Proof. As in the previous lemma, it is sufficient to consider the case $i = 1$ only. It follows from Theorem 2 in [9] that for $\lambda_k \geq 0, k = 1, 2, \dots, N$

$$\begin{aligned} \lim_{m \rightarrow \infty} m \left(1 - \mathbf{E} \left[\exp \left\{ - \sum_{k=1}^N \lambda_k \frac{Z_k(m)}{m^k} \right\} \right] \right) \\ = \lim_{m \rightarrow \infty} m (1 - H_m^{(1,N)}(e^{-\lambda_1/m}, e^{-\lambda_2/m^2}, \dots, e^{-\lambda_N/m^N})) \\ = \Phi(\lambda_1, \lambda_2, \dots, \lambda_N), \end{aligned}$$

where $\Phi = \Phi(\lambda_1, \lambda_2, \dots, \lambda_N)$ solves the differential equation

$$\sum_{k=1}^N k \lambda_k \frac{\partial \Phi}{\partial \lambda_k} = -b_1 \Phi^2 + \Phi + \sum_{k=2}^N f_k \lambda_k$$

with the initial conditions

$$\Phi(\mathbf{0}) = 0, \quad \frac{\partial \Phi(\mathbf{0})}{\partial \lambda_1} = 1, \quad \frac{\partial \Phi(\mathbf{0})}{\partial \lambda_k} = \frac{1}{k-1} f_k, \quad k = 2, \dots, N$$

and

$$f_k = \frac{1}{(k-2)!} \prod_{j=1}^{k-1} m_{j,j+1}, \quad k = 2, \dots, N.$$

Since $m^{2^{k-1}} = m^k$ for $k = 1, 2$ and $m^{2^{k-1}} \gg m^k$ for $k > 2$, we conclude by the continuity of Φ at point $\mathbf{0}$ that

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{\gamma_1} Q_m^{(1,N)}(\mathbf{s}) &= y^{-1} \lim_{m \rightarrow \infty} m Q_m^{(1,N)}(\mathbf{s}) \\ &= y^{-1} \lim_{m \rightarrow \infty} m \left(1 - \mathbf{E} \left[\exp \left\{ - \sum_{k=1}^N \lambda_k \frac{Z_k(m)}{n^{1/2^{N-k}}} \right\} \right] \right) \\ &= y^{-1} \lim_{m \rightarrow \infty} m \left(1 - \mathbf{E} \left[\exp \left\{ - \sum_{k=1}^N \lambda_k y^{2^{k-1}} \frac{Z_1(m)}{m^{2^{k-1}}} \right\} \right] \right) \\ &= y^{-1} \Phi(\lambda_1 y, \lambda_2 y^2, 0, \dots, 0) = y^{-1} \phi_1(\lambda_1 y, \lambda_2 y^2). \end{aligned}$$

Lemma 9 is proved.

Lemma 10 *Let condition (22) be valid. If, for some $i \leq N-1$*

$$n^{\gamma_i} \ll m \ll n^{\gamma_{i+1}} \tag{25}$$

then

$$\lim_{n \rightarrow \infty} n^{\gamma_1} Q_m^{(1,N)}(\mathbf{s}) = D_i(\lambda_{i+1})^{1/2^i}.$$

Proof. It follows from (19) and (25) that

$$\mathbf{P}(T_i > m) \sim c_{1i} m^{-2^{-(i-1)}} = o(n^{-\gamma_1}).$$

Therefore,

$$\begin{aligned} Q_m^{(1,N)}(\mathbf{s}) &= \mathbf{E} \left[1 - s_1^{Z_1(m)} s_2^{Z_2(m)} \dots s_N^{Z_N(m)} \right] \\ &= \mathbf{E} \left[\left(1 - s_{i+1}^{Z_{i+1}(m)} \dots s_N^{Z_N(m)} \right); T_i \leq m \right] + o(n^{-\gamma_1}) \\ &= 1 - H_m^{(1,N)} \left(\mathbf{1}^{(i)}, s_{i+1}, \dots, s_N \right) + o(n^{-\gamma_1}). \end{aligned}$$

It is not difficult to check that for our decomposable branching process

$$\begin{aligned}
& H_m^{(1,N)} \left(\mathbf{1}^{(i)}, s_{i+1}, \dots, s_N \right) \\
&= \mathbf{E} \left[\prod_{k=0}^{m-1} \prod_{r=1}^i \prod_{l=1}^{Z_r(k)} \prod_{j=i+1}^N \left(H_{m-k}^{(j,N)}(\mathbf{s}) \right)^{\eta_{rj}(k,l)} \right] \\
&= \mathbf{E} \left[\prod_{k=0}^{m-1} \prod_{r=1}^i \prod_{l=1}^{Z_r(k)} \prod_{j=i+1}^N \left(H_{m-k}^{(j,N)}(\mathbf{s}) \right)^{\eta_{rj}(k,l)} ; T_i \leq \sqrt{mn^{\gamma_i}} \right] \\
&\quad + O \left(\mathbf{P} \left(T_i > \sqrt{mn^{\gamma_i}} \right) \right).
\end{aligned}$$

Observing that $\lim_{m \rightarrow \infty} H_{m-k}^{(j,N)}(\mathbf{s}) \rightarrow 1$ for $j \geq i+1$ and $k \leq T_i \leq \sqrt{mn^{\gamma_i}} = o(m)$, we get on the set $T_i \leq \sqrt{mn^{\gamma_i}}$

$$\begin{aligned}
& \prod_{k=0}^{m-1} \prod_{r=1}^i \prod_{l=1}^{Z_r(k)} \prod_{j=i+1}^N \left(H_{m-k}^{(j,N)}(\mathbf{s}) \right)^{\eta_{rj}(k,l)} \\
&= \exp \left\{ - \sum_{r=1}^i \sum_{k=0}^{T_i} \sum_{l=1}^{Z_r(k)} \sum_{j=i+1}^N \eta_{rj}(k,l) Q_{m-k}^{(j,N)}(\mathbf{s}) (1 + o(1)) \right\}.
\end{aligned}$$

If $j \geq i+1$ then Lemma 8 and the estimates $m \ll n^{\gamma_{i+1}} \leq n^{\gamma_j}$ yield

$$Q_{m-k}^{(j,N)}(\mathbf{s}) \sim Q_m^{(j,N)}(\mathbf{s}) \sim \lambda_j n^{-\gamma_j}.$$

Hence it follows that on the set $T_i \leq \sqrt{mn^{\gamma_i}} = o(m) = o(n^{\gamma_{i+1}})$

$$\begin{aligned}
& \sum_{r=1}^i \sum_{k=0}^{T_i} \sum_{l=1}^{Z_r(k)} \sum_{j=i+1}^N \eta_{rj}(k,l) Q_{m-k}^{(j,N)}(\mathbf{s}) \\
&= (1 + o(1)) \sum_{j=i+1}^N Q_m^{(j,N)}(\mathbf{s}) \sum_{r=1}^i \sum_{k=0}^{T_i} \sum_{l=1}^{Z_r(k)} \eta_{rj}(k,l) \\
&= (1 + o(1)) \sum_{j=i+1}^N W_{1ij} Q_m^{(j,N)}(\mathbf{s}) \\
&= (1 + o(1)) W_{1i,i+1} Q_m^{(i+1,N)}(\mathbf{s}) + O \left(Q_m^{(i+2,N)}(\mathbf{s}) \right) \sum_{j=i+2}^N W_{1ij} \\
&= (1 + o(1)) W_{1i,i+1} \lambda_{i+1} n^{-\gamma_{i+1}} + O_n(n^{-\gamma_{i+2}} W_{1i}).
\end{aligned}$$

Using the estimates

$$\begin{aligned}
0 &\leq \mathbf{E} \left[\exp \left\{ -(1 + o(1)) W_{1i,i+1} \lambda_{i+1} n^{-\gamma_{i+1}} \right\} \right] \\
&\quad - \mathbf{E} \left[\exp \left\{ -(1 + o(1)) W_{1i,i+1} \lambda_{i+1} n^{-\gamma_{i+1}} - O(n^{-\gamma_{i+2}} W_{1i}) \right\} \right] \\
&\leq 1 - \mathbf{E} \left[\exp \left\{ -O(n^{-\gamma_{i+2}} W_{1i}) \right\} \right] = O \left((n^{-\gamma_{i+2}})^{1/2^i} \right) = O(n^{-\gamma_2})
\end{aligned}$$

where, for the penultimate equality we applied (21), we conclude by (20) that

$$\begin{aligned}
& 1 - H_m^{(1,N)} \left(\mathbf{1}^{(i)}, s_{i+1}, \dots, s_N \right) \\
&= (1 + o(1)) \mathbf{E} \left[1 - \exp \left\{ -(1 + o(1)) W_{1,i+1} \lambda_{i+1} n^{-\gamma_{i+1}} \right\} \right] \\
&\quad + O \left(\mathbf{P} \left(T_i > \sqrt{mn^{\gamma_i}} \right) \right) \\
&= (1 + o(1)) D_i \left(\lambda_{i+1} n^{-\gamma_{i+1}} \right)^{1/2^i} + o(n^{-\gamma_1}) \sim D_i(\lambda_{i+1})^{1/2^i} n^{-\gamma_1}
\end{aligned}$$

as desired.

Lemma 11 *If $m \sim yn^{\gamma_i}$ for some $i \in \{2, 3, \dots, N-1\}$ then*

$$\lim_{n \rightarrow \infty} n^{\gamma_1} Q_m^{(1,N)}(\mathbf{s}) = D_{i-1}(y^{-1} \phi_i(\lambda_i y, \lambda_{i+1} y^2))^{1/2^{i-1}}.$$

Proof. If $m \sim yn^{\gamma_i}$ and $j \geq i$ then $n^{\gamma_j} \sim (y^{-1}m)^{2^{j-i}}$ and, therefore,

$$s_j = \exp \left\{ -\lambda_j n^{-\gamma_j} \right\} = \exp \left\{ -(1 + o(1)) \lambda_j y^{2^{j-i}} m^{-2^{j-i}} \right\}.$$

Hence we may apply Lemma 9 to get, as $n \rightarrow \infty$

$$n^{\gamma_i} Q_m^{(i,N)}(\mathbf{s}) \sim y^{-1} m Q_m^{(i,N)}(s_i, s_{i+1}, \dots, s_N) \sim y^{-1} \phi_i(\lambda_i y, \lambda_{i+1} y^2).$$

Further, as in the previous lemma we have

$$Q_m^{(1,N)}(\mathbf{s}) = 1 - H_m^{(1,N)} \left(\mathbf{1}^{(i-1)}, s_i, \dots, s_N \right) + o(n^{-\gamma_1})$$

and on the set $T_{i-1} \leq \sqrt{mn^{\gamma_{i-1}}} \ll m \sim yn^{\gamma_i}$

$$\begin{aligned}
& \sum_{r=1}^{i-1} \sum_{k=0}^{T_{i-1}} \sum_{l=1}^{Z_r(k)} \sum_{j=i}^N \eta_{rj}(k, l) Q_{m-k}^{(j,N)}(\mathbf{s}) \\
&= (1 + o(1)) \sum_{j=i}^N W_{1,i-1,j} Q_m^{(j,N)}(\mathbf{s}) \\
&= (1 + o(1)) W_{1,i-1,i} Q_m^{(i,N)}(\mathbf{s}) + O \left(Q_m^{(i+1,N)}(\mathbf{s}) \right) \sum_{j=i+1}^N W_{1,i-1,j} \\
&= (1 + o(1)) W_{1,i-1,i} (y^{-1} \phi_i(\lambda_i y, \lambda_{i+1} y^2))^{1/2^{i-1}} n^{-\gamma_{i+1}} \\
&\quad + O(n^{-\gamma_{i+2}} W_{1,i-1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& 1 - H_m^{(1,N)} \left(\mathbf{1}^{(i-1)}, s_i, \dots, s_N \right) \\
&= \mathbf{E} \left[1 - \exp \left\{ -(1 + o(1)) W_{1,i-1,i} y^{-1} \phi_i(\lambda_i y, \lambda_{i+1} y^2) n^{-\gamma_i} \right\} \right] \\
&\quad + O \left(\mathbf{P} \left(T_{i-1} \geq \sqrt{mn^{\gamma_{i+1}}} \right) \right) \\
&= (1 + o(1)) D_{i-1} \left(y^{-1} \phi_i(\lambda_i y, \lambda_{i+1} y^2) n^{-\gamma_i} \right)^{1/2^{i-1}} + o(n^{-\gamma_1}) \\
&\sim D_{i-1} (y^{-1} \phi_i(\lambda_i y, \lambda_{i+1} y^2))^{1/2^{i-1}} n^{-\gamma_1}.
\end{aligned}$$

The lemma is proved.

Lemma 12 For all $i = 1, 2, \dots, N-1$

$$C_N = C_i(m_{i,i+1}b_i c_{i+1,N})^{1/2^i} = D_i(c_{i+1,N})^{1/2^i}. \quad (26)$$

Proof. Using (7) we have

$$c_{iN} = \sqrt{b_i^{-1} m_{i,i+1} c_{i+1,N}} = b_i^{-1} \sqrt{b_i m_{i,i+1} c_{i+1,N}} = c_{ii} \sqrt{b_i m_{i,i+1} c_{i+1,N}}$$

leading in view of (8) and (18) to

$$\begin{aligned} C_N &= c_{1N} = \left(\frac{1}{b_N}\right)^{1/2^{N-1}} \prod_{j=1}^{N-1} \left(\frac{m_{j,j+1}}{b_j}\right)^{1/2^j} = \\ &= c_{1i}(b_i m_{i,i+1})^{1/2^i} \left(\left(\frac{1}{b_N}\right)^{1/2^{N-i}} \prod_{j=i+1}^{N-1} \left(\frac{m_{j,j+1}}{b_j}\right)^{1/2^{j-i}}\right)^{1/2^i} \\ &= c_{1i}(b_i m_{i,i+1} c_{i+1,N})^{1/2^i} = D_i(c_{i+1,N})^{1/2^i} \end{aligned}$$

as desired.

3 Properties of the limiting processes

In this section we give a more detailed description of the properties of the limiting processes. It follows from the definition of $\mathbf{R}(t)$ that if

$$\mathbf{S}_i = (s_{i1}, s_{i2}, \dots, s_{iN}) \in [0, 1]^N \text{ and } t_i \in [\gamma_{i-1}, \gamma_i), i = 1, 2, \dots, N,$$

then

$$\mathbf{E} \left[\prod_{i=1}^N \mathbf{S}_i^{\mathbf{R}(t_i)} \right] = \Omega_N(s_{11}, s_{22}, \dots, s_{NN}),$$

where $\Omega_1(s) = s$ and

$$\Omega_{i+1}(s_1, s_2, \dots, s_{i+1}) = s_1 \left(1 - \sqrt{1 - \Omega_i(s_2, \dots, s_{i+1})}\right), i = 1, 2, \dots \quad (27)$$

If now some intervals $[\gamma_{i-1}, \gamma_i)$ contain more than one point of observation over the process $\mathbf{R}(\cdot)$, say, $\gamma_{i-1} \leq t_{i1} < t_{i2} < \dots < t_{ik_i} < \gamma_i, i = 1, 2, \dots, N$, and ${}_j \mathbf{S}_i = ({}_j s_{i1}, {}_j s_{i2}, \dots, {}_j s_{iN}) \in [0, 1]^N$ then, clearly,

$$\mathbf{E} \left[\prod_{i=1}^N \prod_{j=1}^{k_i} ({}_j \mathbf{S}_i)^{\mathbf{R}(t_{ij})} \right] = \Omega_N \left(\prod_{j=1}^{k_1} {}_j s_{11}, \prod_{j=1}^{k_2} {}_j s_{22}, \dots, \prod_{j=1}^{k_N} {}_j s_{NN} \right).$$

To describe the characteristics of the processes $\mathbf{U}_i(\cdot)$, $i = 1, \dots, N-1$, let, for $(s_i, s_{i+1}) \in [0, 1]^2$

$$\varphi_i(y; s_i, s_{i+1}) = \sqrt{1-s_{i+1}} \frac{(1-s_i) + \sqrt{1-s_{i+1}} \tanh(b_i c_{iN} y \sqrt{1-s_{i+1}})}{(1-s_i) \tanh(b_i c_{iN} y \sqrt{1-s_{i+1}}) + \sqrt{1-s_{i+1}}} \quad (28)$$

with the natural agreement $\varphi_i(y; 1, 1) = 0$ and

$$\varphi_i(y; s_i, 1) = \frac{1-s_i}{b_i c_{iN} y (1-s_i) + 1}.$$

Denote

$$\begin{aligned} X_i(y; \mathbf{s}) &= X_i(y; s_i, s_{i+1}) = \mathbf{E} \left[\mathbf{s}^{\mathbf{U}_i(y)} | \mathbf{U}_i(0) = \mathbf{e}_i \right] \\ &= \mathbf{E} \left[s_i^{U_{ii}(y)} s_{i+1}^{U_{i,i+1}(y)} | \mathbf{U}_i(0) = \mathbf{e}_i \right] \end{aligned}$$

and set

$$\bar{X}_{R_i}(y; \mathbf{s}) = \bar{X}_{R_i}(y; s_i, s_{i+1}) = \mathbf{E}_{R_i} \left[\mathbf{s}^{\mathbf{U}_i(y)} \right] = \mathbf{E}_{R_i} \left[s_i^{U_{ii}(y)} s_{i+1}^{U_{i,i+1}(y)} \right],$$

where the symbol $\mathbf{E}_{R_i}[\cdot]$ means that the process starts by a random number of type i particles distributed as R_i in (9).

It follows from the description of the branching mechanism for $\mathbf{U}_i(\cdot)$ and the general theory of branching processes (see, for instance, [1], p. 201) that $X_i(y; s_i, s_{i+1})$ solves the differential equation

$$\begin{aligned} \frac{\partial}{\partial y} X_i(y; s_i, s_{i+1}) &= 2b_i c_{iN} \left(\frac{1}{2} X_i^2(y; s_i, s_{i+1}) - X_i(y; s_i, s_{i+1}) + \frac{1}{2} s_{i+1} \right), \\ X_i(0; s_i, s_{i+1}) &= s_i. \end{aligned}$$

Direct calculations show that

$$X_i(y; s_i, s_{i+1}) = 1 - \varphi_i(y; s_i, s_{i+1}) \quad (29)$$

and, as a result

$$\bar{X}_{R_i}(y; s_i, s_{i+1}) = 1 - (\varphi_i(y; s_i, s_{i+1}))^{1/2^{i-1}}. \quad (30)$$

One may check by (28) and (30) that

$$\lim_{y \downarrow 0} \bar{X}_{R_i}(y; s_i, s_{i+1}) = 1 - (1-s_i)^{1/2^{i-1}} \quad (31)$$

and

$$\lim_{y \uparrow \infty} \bar{X}_{R_i}(y; s_i, s_{i+1}) = 1 - (1-s_{i+1})^{1/2^i}. \quad (32)$$

For $y_k \in [0, \infty)$, $(s_{ki}, s_{k,i+1}) \in [0, 1]^2$, $k = 1, 2, \dots, p$; $i = 1, \dots, N-1$ denote $\mathbf{y}_{l,p} = (y_l, \dots, y_p)$ and $\mathbf{S}_{l,p}^{(i)} = (s_{li}, s_{l,i+1}, s_{l+1,i}, s_{l+1,i+1}, \dots, s_{pi}, s_{p,i+1})$.

Using (29) set

$$X_i^{(2)}(\mathbf{y}_{1,2}; \mathbf{S}_{1,2}^{(i)}) = X_i(y_1; s_{1i} X_i(y_2; s_{2i}, s_{2,i+1}), s_{1,i+1} s_{2,i+1})$$

and, by induction

$$X_i^{(p)}(\mathbf{y}_{1,p}; \mathbf{S}_{1,p}^{(i)}) = X_i\left(y_1; s_{1i} X_i^{(p-1)}(\mathbf{y}_{2,p}; \mathbf{S}_{2,p}^{(i)}), \prod_{r=1}^p s_{r,i+1}\right).$$

Finally, recalling (30) put

$$\bar{X}_{R_i}(\mathbf{y}_{1,p}; \mathbf{S}_{1,p}^{(i)}) = 1 - \left(1 - X_i^{(p)}(\mathbf{y}_{1,p}; \mathbf{S}_{1,p}^{(i)})\right)^{1/2^{i-1}}.$$

It is not difficult to check that

$$\bar{X}_{R_i}(\mathbf{y}_{1,p}; \mathbf{S}_{1,p}^{(i)}) = \mathbf{E}_{R_i} \left[s_{1i}^{U_{ii}(y_1)} s_{1,i+1}^{U_{i,i+1}(y_1)} \dots s_{pi}^{U_{ii}(y_p)} s_{p,i+1}^{U_{i,i+1}(y_p)} \right].$$

To complete the description of the limiting processes we are interesting in introduce the function

$$\psi(x; s) = \frac{1}{x + (1-x)/(1-s)}, \quad s \in [0, 1], x \in [0, 1],$$

and consider an N -dimensional process $\mathbf{U}_N(\cdot) = (U_{N1}(\cdot), \dots, U_{NN}(\cdot))$ in which the first $N-1$ components are equal to zero while $U_{NN}(\cdot)$ may be obtained by a time-change from the following single-type continuous time Markov process $\sigma(t), 0 \leq t < \infty$. The life-length distribution of particles in $\sigma(\cdot)$ is exponential with parameter 1. Dying each particle produces exactly two children. One may check (compare, for instance, with Example 3, Section 8, Chapter 1 in [14]) that

$$\mathbf{E} \left[s^{\sigma(t)} | \sigma(0) = 1 \right] = 1 - \psi(1 - e^{-t}; s).$$

Assuming that $\sigma(0) \stackrel{d}{=} R_N$ (recall (10)) and making the change of time $x = 1 - e^{-t}, 0 \leq t < \infty$, we obtain an inhomogeneous single-type branching process, denoted by $U_{NN}(\cdot)$ such that

$$\bar{G}_{R_N}(x; s) = \mathbf{E}_{R_N} \left[s^{U_{NN}(x)} \right] = 1 - (\psi(x; s))^{1/2^{N-1}}$$

and

$$\mathbf{E} \left[s^{U_{NN}(x+\Delta)} | U_{NN}(x) = 1 \right] = 1 - \psi\left(\frac{\Delta}{1-x}; s\right), \quad 0 < x + \Delta < 1.$$

Let, further, for $x_j \in [0, 1]$ and $\mathbf{S}_{j,p} = (s_j, \dots, s_p), j = 1, 2, \dots, p$

$$G^{(1)}(x_1; s_1) = G(x_1; s_1) = 1 - \psi(x_1; s_1)$$

and, by induction

$$G^{(p)}(\mathbf{x}_{1,p}; \mathbf{S}_{1,p}) = G\left(x_1; s_{1N} G^{(p-1)}\left(\frac{\mathbf{x}_{2,p}}{1-x_1}; \mathbf{S}_{2,p}\right)\right).$$

One may check that

$$\begin{aligned} \bar{G}_{R_N}(\mathbf{x}_{1,p}; \mathbf{S}_{1,p}) &= \mathbf{E}_{R_N} \left[s_1^{U_{NN}(x_1)} s_2^{U_{NN}(x_2)} \dots s_p^{U_{NN}(x_p)} \right] \\ &= 1 - (1 - G^{(p)}(\mathbf{x}_{1,p}; \mathbf{S}_{1,p}))^{1/2^{N-1}}. \end{aligned}$$

4 Convergence of one-dimensional distributions

As the first step in proving the main results of the paper we establish convergence of one-dimensional distributions of $\{\mathbf{Z}(m, n), 0 \leq m \leq n\}$ given $\mathbf{Z}(n) \neq \mathbf{0}$. Let

$$H_{m,n}^{(k,N)}(\mathbf{s}) = \mathbf{E} \left[\mathbf{s}^{\mathbf{Z}(m,n)} | \mathbf{Z}(0) = \mathbf{e}_k \right], J_{m,n}^{(k,N)}(\mathbf{s}) = \mathbf{E} \left[\mathbf{s}^{\mathbf{Z}(m,n)} | \mathbf{Z}(n) \neq \mathbf{0}, \mathbf{Z}(0) = \mathbf{e}_k \right],$$

$$\mathbf{H}_{m,n}(\mathbf{s}) = \left(H_{m,n}^{(1,N)}(\mathbf{s}), \dots, H_{m,n}^{(N,N)}(\mathbf{s}) \right), \mathbf{J}_{m,n}(\mathbf{s}) = \left(J_{m,n}^{(1,N)}(\mathbf{s}), \dots, J_{m,n}^{(N,N)}(\mathbf{s}) \right).$$

For $\mathbf{x} = (x_1, \dots, x_N)$ and $\mathbf{y} = (y_1, \dots, y_N)$ put $\mathbf{x} \otimes \mathbf{y} = (x_1 y_1, x_2 y_2, \dots, x_N y_N)$ and denote

$$\begin{aligned} s'_k &= s_k Q_{n-m}^{(k,N)} + (1 - Q_{n-m}^{(k,N)}) = 1 - (1 - s_k) Q_{n-m}^{(k,N)}, \\ \mathbf{s}' &= (s'_1, \dots, s'_N) = \mathbf{1} - (\mathbf{1} - \mathbf{s}) \otimes \mathbf{Q}_{n-m}. \end{aligned} \tag{33}$$

It is not difficult to understand that

$$H_{m,n}^{(k,N)}(\mathbf{s}) = H_m^{(k,N)}(\mathbf{s}') = H_m^{(k,N)}(\mathbf{1} - (\mathbf{1} - \mathbf{s}) \otimes \mathbf{Q}_{n-m})$$

and that

$$J_{m,n}^{(k,N)}(\mathbf{s}) = \mathbf{E} \left[\mathbf{s}^{\mathbf{Z}(m,n)} | \mathbf{Z}(n) \neq \mathbf{0}, \mathbf{Z}(0) = \mathbf{e}_k \right] = 1 - \frac{Q_m^{(k,N)}(\mathbf{s}')}{Q_n^{(k,N)}}. \tag{34}$$

Theorem 13 *Let Hypothesis A be valid.*

1) *If $m \ll n^{\gamma_1}$ then*

$$\lim_{n \rightarrow \infty} J_{m,n}^{(1,N)}(\mathbf{s}) = \lim_{n \rightarrow \infty} \mathbf{E} \left[\mathbf{s}^{\mathbf{Z}(m,n)} | \mathbf{Z}(n) \neq \mathbf{0}, \mathbf{Z}(0) = \mathbf{e}_1 \right] = s_1. \tag{35}$$

2) *If $n^{\gamma_i} \ll m \ll n^{\gamma_{i+1}}$ for some $i \in \{1, 2, \dots, N-1\}$ then*

$$\lim_{n \rightarrow \infty} J_{m,n}^{(1,N)}(\mathbf{s}) = 1 - (1 - s_{i+1})^{1/2^i}. \tag{36}$$

3) If $m = (y + l_n)n^{\gamma_i}$, $y \in [0, \infty)$ for some $i \in \{1, 2, \dots, N-1\}$ then

$$\lim_{n \rightarrow \infty} J_{m,n}^{(1,N)}(\mathbf{s}) = \bar{X}_{R_i}(y; s_i, s_{i+1}). \quad (37)$$

4) If $m = (x + l_n)n$, $x \in [0, 1)$ then

$$\lim_{n \rightarrow \infty} J_{m,n}^{(1,N)}(\mathbf{s}) = \bar{G}_{R_N}(x; s_N). \quad (38)$$

Proof. We start by observing that if $m \ll n$ then

$$\begin{aligned} 1 - s'_i &= (1 - s_i)Q_{n-m}^{(i,N)} \sim (1 - s_i)Q_n^{(i,N)} \\ &\sim 1 - \exp\left\{-(1 - s_i)Q_n^{(i,N)}\right\} \sim 1 - \exp\left\{-(1 - s_i)c_{iN}n^{-\gamma_i}\right\}. \end{aligned}$$

This representation allows us to use the previous results with s_i and λ_i replaced by s'_i and $(1 - s_i)c_{iN}$, respectively.

Recalling (15) and applying Lemma 8 we get

$$\lim_{n \rightarrow \infty} \frac{Q_m^{(1,N)}(\mathbf{s}')}{Q_n^{(1,N)}} = \lim_{n \rightarrow \infty} n^{2^{-(N-1)}} \frac{1 - H_m^{(1,N)}(\mathbf{s}')}{C_N} = 1 - s_1.$$

Hence (35) follows.

Applying Lemma 10 with $n^{\gamma_i} \ll m \ll n^{\gamma_{i+1}}$ and recalling Lemma 12 we conclude

$$\lim_{n \rightarrow \infty} \frac{Q_m^{(1,N)}(\mathbf{s}')}{Q_n^{(1,N)}} = \frac{D_i}{C_N} ((1 - s_{i+1})c_{i+1,N})^{1/2^i} = (1 - s_{i+1})^{1/2^i}$$

leading to (36).

Proof of (37). If $y = 0$ then the needed statement follows from (35) and (36). If $i \in \{1, 2, \dots, N-1\}$ is fixed and $m \sim yn^{\gamma_i}$, $y > 0$, then for $j \geq i$

$$\begin{aligned} 1 - s'_j &\sim 1 - \exp\left\{-(1 - s_j)c_{jN}n^{-\gamma_j}\right\} \\ &\sim 1 - \exp\left\{-(1 - s_j)c_{jN}y^{2^{j-i}}m^{-2^{j-i}}\right\}. \end{aligned}$$

Hence, by (15) and Lemmas 9 and 11 we get

$$\lim_{n \rightarrow \infty} \frac{Q_m^{(1,N)}(\mathbf{s}')}{Q_n^{(1,N)}} = \frac{D_{i-1}}{C_N} \left(\frac{\phi_i(c_{iN}(1 - s_i)y, c_{i+1,N}(1 - s_{i+1})y^2)}{y} \right)^{1/2^{i-1}}$$

where we agree to write $D_0 = 1$. By (24) and (7)

$$\begin{aligned}
& \frac{\phi_i(c_{iN}(1-s_i)y, c_{i+1,N}(1-s_{i+1})y^2)}{y} \\
&= \sqrt{\frac{m_{i,i+1}c_{i+1,N}(1-s_{i+1})}{b_i}} \times \\
& \quad \times \frac{b_i c_{iN}(1-s_i) + \sqrt{b_i m_{i,i+1} c_{i+1,N}(1-s_{i+1})} \tanh y \sqrt{b_i m_{i,i+1} c_{i+1,N}(1-s_{i+1})}}{b_i c_{iN}(1-s_i) \tanh y \sqrt{b_i m_{i,i+1} c_{i+1,N}(1-s_{i+1})} + \sqrt{b_i m_{i,i+1} c_{i+1,N}(1-s_{i+1})}} \\
&= c_{iN} \sqrt{1-s_{i+1}} \times \frac{b_i c_{iN}(1-s_i) + b_i c_{iN} \sqrt{1-s_{i+1}} \tanh(y b_i c_{iN} \sqrt{1-s_{i+1}})}{b_i c_{iN}(1-s_i) \tanh(y b_i c_{iN} \sqrt{1-s_{i+1}}) + b_i c_{iN} \sqrt{1-s_{i+1}}} \\
&= c_{iN} \sqrt{1-s_{i+1}} \times \frac{1-s_i + \sqrt{1-s_{i+1}} \tanh(y b_i c_{iN} \sqrt{1-s_{i+1}})}{(1-s_i) \tanh(y b_i c_{iN} \sqrt{1-s_{i+1}}) + \sqrt{1-s_{i+1}}}.
\end{aligned}$$

To complete the proof of (37) it remains to recall (26).

Proof of (38). If $x = 0$ then (38) follows from (36). Consider now the case $m \sim xn, 0 < x < 1$. Observe that for $\mathbf{s} = (s_1, s_2, \dots, s_N) \in [0, 1]^N$

$$\begin{aligned}
H_m^{(1,N)}(\mathbf{1}^{(N-1)}, s_N) - H_m^{(1,N)}(\mathbf{s}) &= \mathbf{E} \left[\left(1 - s_1^{Z_1(m)} \dots s_{N-1}^{Z_{N-1}(m)} \right) s_N^{Z_N(m)} \right] \\
&\leq \mathbf{E} \left[1 - s_1^{Z_1(m)} \dots s_{N-1}^{Z_{N-1}(m)} \right] \leq \mathbf{P}(T_{N-1} > m) \leq cm^{-2^{-(N-2)}}. \quad (39)
\end{aligned}$$

Thus,

$$1 - H_m^{(1,N)}(\mathbf{s}) = 1 - H_m^{(1,N)}(\mathbf{1}^{(N-1)}, s_N) + \varepsilon_{m,n}(\mathbf{s}) Q_m^{(1,N)}$$

where $\varepsilon_{m,n}(\mathbf{s}) \rightarrow 0$ as $n \rightarrow \infty, m \sim xn$ uniformly in $\mathbf{s} \in [0, 1]^N$. Therefore,

$$1 - H_m^{(1,N)}(\mathbf{s}') = 1 - H_m^{(1,N)}(\hat{\mathbf{s}}, 1 - (1 - s_N) Q_{n-m}^{(N,N)}) + \varepsilon'_{m,n}(\mathbf{s}) Q_n^{(1,N)}$$

where $\varepsilon'_{m,n}(\mathbf{s}) \rightarrow 0$ as $n \rightarrow \infty, m \sim xn$ uniformly in $\hat{\mathbf{s}} = (s'_1, \dots, s'_{N-1}) \in [0, 1]^{N-1}$.

We now select an integer $r = r(m, n) \in \mathbb{N}^* = \{1, 2, \dots\}$ in such a way that

$$H_{r-1}^{(N,N)}(0) \leq 1 - (1 - s_N) Q_{n-m}^{(N,N)} \leq H_r^{(N,N)}(0)$$

or

$$Q_r^{(N,N)} = 1 - H_r^{(N,N)}(0) \leq (1 - s_N) Q_{n-m}^{(N,N)} \leq Q_{r-1}^{(N,N)} = 1 - H_{r-1}^{(N,N)}(0).$$

This is possible, since by (15)

$$Q_{n-m}^{(N,N)} \sim \frac{1}{(n-m)b_N} \rightarrow 0, \quad n-m \rightarrow \infty. \quad (40)$$

In particular,

$$r \sim \frac{n-m}{1-s_N}. \quad (41)$$

Under our choice of r , for any $\hat{\mathbf{s}} \in [0, 1]^{N-1}$

$$H_m^{(1,N)}(\hat{\mathbf{s}}, H_{r-1}^{(N,N)}(0)) \leq H_m^{(1,N)}(\hat{\mathbf{s}}, 1 - (1 - s_N)Q_{n-m}^{(N,N)}) \leq H_m^{(1,N)}(\hat{\mathbf{s}}, H_r^{(N,N)}(0)).$$

Letting $\hat{\mathbf{s}} = (H_r^{(1,N)}(0), \dots, H_r^{(N-1,N)}(0))$ we get by the branching property of generating functions the estimate

$$H_m^{(1,N)}(\hat{\mathbf{s}}, 1 - (1 - s_N)Q_{n-m}^{(N,N)}) \leq H_m^{(1,N)}(\mathbf{H}_r(0)) = H_{m+r}^{(1,N)}(0)$$

implying

$$1 - H_m^{(1,N)}(\mathbf{s}') \geq 1 - H_{m+r}^{(1,N)}(0) + \varepsilon'_{m,n}Q_n^{(1,N)} = Q_{m+r}^{(1,N)} + \varepsilon'_{m,n}Q_n^{(1,N)},$$

where $\varepsilon'_{m,n} \rightarrow 0$ as $n \rightarrow \infty$, $m \sim xn$, while $\hat{\mathbf{s}} = (H_{r-1}^{(1,N)}(0), \dots, H_{r-1}^{(N-1,N)}(0))$ gives the inequality

$$H_m^{(1,N)}(\hat{\mathbf{s}}, 1 - (1 - s_N)Q_{n-m}^{(N,N)}) \geq H_m^{(1,N)}(\mathbf{H}_r(0)) = H_{m+r-1}^{(1,N)}(0)$$

leading in the range under consideration to

$$1 - H_m^{(1,N)}(\mathbf{s}') \leq Q_{m+r-1}^{(1,N)} + \varepsilon'_{m,n}Q_n^{(1,N)}.$$

Hence

$$1 - H_m^{(1,N)}(\mathbf{s}') = Q_{m+r}^{(1,N)} + \varepsilon''_{m,n}Q_n^{(1,N)}$$

where $\varepsilon''_{m,n} \rightarrow 0$ as $n \rightarrow \infty$, $m \sim xn$. We now conclude by (15) that

$$1 - H_m^{(1,N)}(\mathbf{s}') \sim Q_{m+r}^{(1,N)} \sim C_N(m+r)^{-2^{-(N-1)}}.$$

Hence, on account of (41) and $m \sim xn, 0 < x < 1$, we get (recall (7))

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1 - H_m^{(1,N)}(\mathbf{s}')}{Q_n^{(1,N)}} &= \lim_{n \rightarrow \infty} \left(\frac{n}{nx + n(1-x)/(1-s_N)} \right)^{2^{-(N-1)}} \\ &= \left(\frac{1}{x + (1-x)/(1-s_N)} \right)^{2^{-(N-1)}} \end{aligned}$$

completing the proof of (38).

Theorem 13 is proved.

Proof of Theorem 6. Since our process is decomposable and strongly critical, it is sufficient to check (16) for $i = 1$ only. For $\hat{s}_N = \exp(-\lambda/(nb_N))$ we have

$$\mathbf{E} \left[\exp \left\{ -\lambda \frac{Z_N(n)}{b_N n} \right\} \middle| \mathbf{Z}(n) \neq \mathbf{0} \right] = 1 - \frac{1 - H_n^{(1,N)}(\mathbf{1}^{(N-1)}, \hat{s}_N)}{Q_n^{(1,N)}}.$$

We now select an integer $r = r(\lambda, n) \in \mathbb{N}^* = \{1, 2, \dots\}$ in such a way that

$$H_{r-1}^{(N,N)}(0) \leq \hat{s}_N \leq H_r^{(N,N)}(0).$$

It follows from (40) that $r \sim n\lambda^{-1}$. Letting $s_i = H_r^{(i,N)}(\mathbf{0})$, $i = 1, 2, \dots, N$, and setting $\mathbf{s} = (s_1, \dots, s_N)$ we get by (39) after evident estimates that

$$\left| H_n^{(1,N)}(\mathbf{1}^{(N-1)}, \hat{s}_N) - H_n^{(1,N)}(\mathbf{s}) \right| \leq cn^{1/2^{N-2}}.$$

Hence, using (15) with $i = 1$ we obtain

$$\begin{aligned} \frac{1 - H_n^{(1,N)}(\mathbf{1}^{(N-1)}, \hat{s}_N)}{Q_n^{(1,N)}} &\sim \frac{1 - H_n^{(1,N)}(\mathbf{s})}{Q_n^{(1,N)}} = \frac{Q_{r+n}^{(1,N)}}{Q_n^{(1,N)}} \\ &\sim \left(\frac{n}{r+n} \right)^{1/2^{N-1}} \sim \left(\frac{\lambda}{1+\lambda} \right)^{1/2^{N-1}} \end{aligned}$$

as desired.

5 Convergence of finite-dimensional distributions

In this section we study the limiting behavior of the finite-dimensional distributions of the reduced process $\{\mathbf{Z}(m, n), 0 \leq m \leq n\}$. Our first theorem deals with the case $m \ll n$.

Theorem 14 *Let Hypothesis A be valid and $\mathbf{S}_l = (s_{l1}, \dots, s_{lN})$, $l = 1, 2, \dots, p$.*

1) *If, for a fixed $i \in \{0, 1, \dots, N-1\}$*

$$n^{\gamma_i} \ll m_l \ll n^{\gamma_{i+1}}, \quad l = 1, \dots, p$$

then

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[\prod_{l=1}^p \mathbf{S}_l^{\mathbf{Z}(m_l, n)} \mid \mathbf{Z}(n) \neq 0 \right] = 1 - \left(1 - \prod_{l=1}^p s_{l, i+1} \right)^{1/2^i}. \quad (42)$$

2) *Let $0 = Y_1 < Y_2 < \dots < Y_p < \infty$ be a tuple of nonnegative numbers with $y_1 = 0$, $y_l = Y_l - Y_{l-1}$, $l = 2, \dots, p$. If, for a fixed $i \in \{1, 2, \dots, N-1\}$*

$$m_1 \sim l_n n^{\gamma_i}, \quad m_l \sim Y_l n^{\gamma_i}, \quad l = 2, \dots, p$$

then

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[\prod_{l=1}^p \mathbf{S}_l^{\mathbf{Z}(m_l, n)} \mid \mathbf{Z}(n) \neq 0 \right] = \bar{X}_{R_i}(\mathbf{y}_{1,p}; \mathbf{S}_{1,p}^{(i)}). \quad (43)$$

The second theorem is devoted to the finite-dimensional distributions of the reduced process when m is of order n .

Theorem 15 *Let Hypothesis A be valid and $0 = X_1 < X_2 < \dots < X_p < 1$ be a tuple of nonnegative numbers with $x_1 = 0$, $x_l = X_l - X_{l-1}$, $l = 2, \dots, p$. If*

$$m_1 \sim l_n n, \quad m_l \sim X_l n, \quad l = 2, \dots, p$$

then

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[\prod_{l=1}^p \mathbf{S}_l^{\mathbf{Z}(m_l, n)} \mid \mathbf{Z}(n) \neq 0 \right] = \tilde{G}_{R_N}(\mathbf{x}_{1,p}; \mathbf{S}_{1,p;N}),$$

where $\mathbf{S}_{1,p;N} = (s_{1N}, s_{2N}, \dots, s_{pN})$.

To prove Theorems 14 and 15 we need additional notation.

For $0 \leq m_0 < m_1 < \dots < m_p \leq n$ set $\mathbf{m} = (m_0, m_1, \dots, m_p)$, put $\Delta_i = m_i - m_{i-1}$, and denote

$$\begin{aligned} \hat{J}_{m_0, m_1, \dots, m_p, n}^{(i, N)}(\mathbf{S}_1, \dots, \mathbf{S}_p) &= \hat{J}_{\mathbf{m}, n}^{(i, N)}(\mathbf{S}_1, \dots, \mathbf{S}_p) \\ &= \mathbf{E} \left[\prod_{l=1}^p \mathbf{S}_l^{\mathbf{Z}(m_l, n)} \mid \mathbf{Z}(m_0, n) = \mathbf{e}_i \right] \end{aligned}$$

and

$$\hat{\mathbf{J}}_{\mathbf{m}, n}(\mathbf{S}_1, \dots, \mathbf{S}_p) = \left(\hat{J}_{\mathbf{m}, n}^{(1, N)}(\mathbf{S}_1, \dots, \mathbf{S}_p), \dots, \hat{J}_{\mathbf{m}, n}^{(N, N)}(\mathbf{S}_1, \dots, \mathbf{S}_p) \right).$$

The next statement is a simple observation following from Corollary 2 in [16].

Lemma 16 *For any $0 \leq m_0 < m_1 < \dots < m_p \leq n$ we have*

$$\begin{aligned} \hat{J}_{\mathbf{m}, n}^{(1, N)}(\mathbf{S}_1, \dots, \mathbf{S}_p) &= \hat{J}_{m_0, m_1, n}^{(1, N)} \left(\mathbf{S}_1 \otimes \hat{\mathbf{J}}_{m_1, m_2, \dots, m_p, n}(\mathbf{S}_2, \dots, \mathbf{S}_p) \right) \\ &= J_{\Delta_1, n-m_0}^{(1, N)} \left(\mathbf{S}_1 \otimes \mathbf{J}_{\Delta_2, n-m_1}(\mathbf{S}_2 \otimes \dots (\mathbf{S}_{p-1} \otimes \mathbf{J}_{\Delta_p, n-m_{p-1}}(\mathbf{S}_p)) \dots) \right). \end{aligned}$$

In particular, if $\mathbf{m} = (0, m_1, m_2)$ then

$$\hat{J}_{\mathbf{m}, n}^{(1, N)}(\mathbf{S}_1, \mathbf{S}_2) = J_{m, n}^{(1, N)}(\mathbf{S}_1 \otimes \mathbf{J}_{\Delta_2, n-m_1}(\mathbf{S}_2))$$

and if $\mathbf{m} = (m_0, m_1)$ then for $\mathbf{s} = (s_1, \dots, s_N)$

$$\hat{J}_{m_0, m_1, n}^{(k, N)}(\mathbf{s}) = J_{\Delta_1, n-m_0}^{(k, N)}(\mathbf{s}) = 1 - \frac{1 - H_{\Delta_1}^{(k, N)}(\mathbf{1} - (\mathbf{1} - \mathbf{s}) \otimes \mathbf{Q}_{n-m_1})}{Q_{n-m_0}^{(k, N)}}. \quad (44)$$

Using (44) we prove the following statement.

Lemma 17 *If $m_0 = (Y_0 + l_n)n^{\gamma_i} < m_1 = (Y_1 + l_n)n^{\gamma_i}$ then for any $j \geq i$ there exists a constant $\chi \in (0, \infty)$ such that for all $n \geq n_0$*

$$\mathbf{P}(\mathbf{Z}(m_1, n) = \mathbf{e}_j \mid \mathbf{Z}(m_0, n) = \mathbf{e}_j) \geq 1 - \chi(Y_1 - Y_0).$$

Proof. By the decomposability assumption and the condition $m_{jj} = 1$ implying

$$m_{jj}(\Delta_1) = \frac{\partial H_{\Delta_1}^{(j,N)}(\mathbf{s})}{\partial s_j} \Big|_{\mathbf{s}=\mathbf{1}} = 1$$

we get

$$1 - \frac{\partial H_{\Delta_1}^{(j,N)}(\mathbf{s})}{\partial s_j} \Big|_{\mathbf{s}=\mathbf{H}_{n-m_1}(\mathbf{0})} \leq \sum_{k=j}^N \mathbf{E} Z_j(\Delta_1) (Z_k(\Delta_1) - \delta_{kj}) Q_{n-m_1}^{(k,N)}.$$

Recalling (14) and (15) and setting $h = Y_1 - Y_0$ we obtain

$$\begin{aligned} \mathbf{E} Z_j(\Delta_1) Z_k(\Delta_1) Q_{n-m_1}^{(k,N)} &\leq c_0 (n - m_1)^{-1/2^{N-k}} (\Delta_1)^{k-j+1} \\ &\leq c_0 (n - m_1)^{-1/2^{N-k}} (hn^{1/2^{N-i}})^{k-j+1} \\ &\leq \chi hn^{-1/2^{N-k}} (n^{1/2^{N-i}})^{k-j+1} \end{aligned}$$

for some constants $0 < c_0 \leq \chi < \infty$. On account of $k \geq j \geq i$ we have

$$\frac{k-j+1}{2^{N-i}} - \frac{1}{2^{N-k}} = \frac{1}{2^{N-i}} (k-j+1 - 2^{k-i}) \leq 0.$$

Thus,

$$1 - \frac{\partial H_{\Delta_1}^{(j,N)}(\mathbf{s})}{\partial s_j} \Big|_{\mathbf{s}=\mathbf{H}_{n-m_1}(\mathbf{0})} \leq \chi h.$$

Hence, using the previous lemma and monotonicity of $Q_r^{(j,N)}$ in r we get

$$\begin{aligned} \mathbf{P}(\mathbf{Z}(m_1, n) = \mathbf{e}_j | \mathbf{Z}(m_0, n) = \mathbf{e}_j) &= \frac{Q_{n-m_1}^{(j,N)} \frac{\partial H_{\Delta_1}^{(j,N)}(\mathbf{s})}{\partial s_j} \Big|_{\mathbf{s}=\mathbf{H}_{n-m_1}(\mathbf{0})}}{Q_{n-m_0}^{(j,N)} \frac{\partial H_{\Delta_1}^{(j,N)}(\mathbf{s})}{\partial s_j} \Big|_{\mathbf{s}=\mathbf{H}_{n-m_1}(\mathbf{0})}} \quad (45) \\ &\geq \frac{\partial H_{\Delta_1}^{(j,N)}(\mathbf{s})}{\partial s_j} \Big|_{\mathbf{s}=\mathbf{H}_{n-m_1}(\mathbf{0})} \geq 1 - \chi h. \end{aligned}$$

Lemma 17 is proved.

Proof of Theorem 14. Using (34) and Theorem 13 we see that

1) if $m \ll n^{\gamma_k}$ then

$$\lim_{n \rightarrow \infty} J_{m,n}^{(k,N)}(\mathbf{s}) = s_k; \quad (46)$$

2) if $m = (y + l_n)n^{\gamma_k} = (y + l_n)n^{1/2^{(N-k)}}$, $y \in [0, \infty)$ then

$$\lim_{n \rightarrow \infty} J_{m,n}^{(k,N)}(\mathbf{s}) = X_k(y; s_k, s_{k+1});$$

3) if $m = (x + l_n)n$, $x \in [0, 1]$ then

$$\lim_{n \rightarrow \infty} J_{m,n}^{(N,N)}(\mathbf{s}) = G(x; s_N). \quad (47)$$

Proof of (42). Consider first the case $p = 2$ and take $\mathbf{m} = (0, m_1, m_2)$. By Lemma 16

$$\hat{J}_{\mathbf{m},n}^{(1,N)}(\mathbf{S}_1, \mathbf{S}_2) = J_{m_1,n}^{(1,N)}(\mathbf{S}_1 \otimes \mathbf{J}_{\Delta_2, n-m_1}(\mathbf{S}_2)). \quad (48)$$

It follows from (36) that, given $n^{\gamma_i} \ll m_1 \ll n^{\gamma_{i+1}}$

$$J_{m_1,n}^{(1,N)}(\mathbf{S}_1) \rightarrow 1 - (1 - s_{1,i+1})^{1/2^i}$$

as $n \rightarrow \infty$. Further, in view of $\Delta_2 = m_2 - m_1 \ll n^{\gamma_{i+1}}$ and (46) $J_{\Delta_2, n-m_1}^{(i+1,N)}(\mathbf{S}_2) \rightarrow s_{2,i+1}$ as $n \rightarrow \infty$. Hence, using the continuity of the functions under consideration and (48) we get

$$\lim_{n \rightarrow \infty} \hat{J}_{\mathbf{m},n}^{(1,N)}(\mathbf{S}_1, \mathbf{S}_2) = 1 - (1 - s_{1,i+1} s_{2,i+1})^{1/2^i}.$$

The validity of (42) for any $p > 3$ may be checked by induction using Lemma 16.

Proof of (43). Consider again the case $p = 2$ only. It follows from (37) that, given $m_l \sim Y_l n^{\gamma_i}$, $l = 1, 2$, with $Y_1 = y_1$

$$J_{m_1,n}^{(1,N)}(\mathbf{s}) \rightarrow \bar{X}_{R_i}(y_1; s_i, s_{i+1})$$

as $n \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} J_{\Delta_2, n-m_1}^{(i,N)}(\mathbf{S}_2) = X_i(y_2; s_{2i}, s_{2,i+1}), \quad \lim_{n \rightarrow \infty} J_{\Delta_2, n-m_1}^{(i+1,N)}(\mathbf{S}_2) = s_{2,i+1}.$$

Hence, using the continuity of the functions involved and (48) we get

$$\lim_{n \rightarrow \infty} \hat{J}_{\mathbf{m},n}^{(1,N)}(\mathbf{S}_1, \mathbf{S}_2) = \bar{X}_{R_i}(y_1; s_{1i} X_i(y_2; s_{2i}, s_{2,i+1}), s_{1,i+1} s_{2,i+1})$$

proving (43) for $p = 2$.

To justify (43) for $p > 3$ it is necessary to use Lemma 16 and induction arguments. We omit the respective details.

Proof of Theorem 15. We consider the case $p = 2$ only and to this aim take $\mathbf{m} = (0, (x_1 + l_n)n, (x_1 + x_2 + l_n)n)$. By (48), (38) and (47)

$$\begin{aligned} \lim_{n \rightarrow \infty} \hat{J}_{\mathbf{m},n}^{(1,N)}(\mathbf{S}_1, \mathbf{S}_2) &= \lim_{n \rightarrow \infty} J_{(x_1+l_n)n,n}^{(1,N)}(\mathbf{S}_1 \otimes \mathbf{J}_{x_2n,n(1-x_1-l_n)}(\mathbf{S}_2)) \\ &= \bar{G}_{R_N} \left(x_1; s_{1N} G \left(\frac{x_2}{1-x_1}; s_{2N} \right) \right) = \bar{G}_{R_N}(\mathbf{x}_{1,2}; \mathbf{S}_{1,2;N}). \end{aligned}$$

The desired statement for $p > 2$ follows by induction.

Proof of point 1) of Theorem 2. Let $0 = t_0 < t_1 < \dots < t_p < 1$. If $\gamma_{i-1} \leq t_1 < t_p < \gamma_i$ for some $i \in \{1, 2, \dots, N\}$ then the needed convergence of finite-dimensional distributions follows from (42). We now consider another extreme case, namely, take a tuple $0 = t_0 < t_1 < \dots < t_N < 1$ such that $\gamma_{i-1} \leq t_i < \gamma_i$ for all $i = 1, 2, \dots, N$. Then for $m_i \sim n^{t_i} g_n(t_i)$ we have

$$n^{\gamma_{i-1}} \ll m_i \ll n^{\gamma_i}, \quad \Delta_i = m_i - m_{i-1} \sim m_i, \quad n - m_i \sim n.$$

These relations, (36), (46), and the continuity of the respective probability generating functions imply (recall (27))

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \hat{J}_{\mathbf{m},n}^{(1,N)}(\mathbf{S}_1, \dots, \mathbf{S}_N) \\
&= \lim_{n \rightarrow \infty} J_{m_1,n}^{(1,N)}(\mathbf{S}_1 \otimes \mathbf{J}_{m_2,n}(\mathbf{S}_2 \otimes \dots (\mathbf{S}_{N-1} \otimes \mathbf{J}_{m_N,n}(\mathbf{S}_N)) \dots)) \\
&= s_{11} \left(1 - \sqrt{1 - \lim_{n \rightarrow \infty} J_{m_2,n}^{(2,N)}(\mathbf{S}_2 \otimes \dots (\mathbf{S}_{N-1} \otimes \mathbf{J}_{m_N,n}(\mathbf{S}_N)) \dots)} \right) \\
&= s_{11} \left(1 - \sqrt{1 - s_{22} \left(1 - \sqrt{1 - \Omega_{N-2}(s_{33}, \dots, s_{NN})} \right)} \right) \\
&= \dots = \Omega_N(s_{11}, s_{22}, \dots, s_{NN})
\end{aligned}$$

as required.

The case when several values among t_j are contained in a subinterval $[\gamma_{i-1}, \gamma_i)$ may be considered by combining the previous arguments. We omit the respective details.

6 Tightness

Denote by $\mathbf{z}^{(i,i+1)}, 1 \leq i \leq N-1$, the $(N-2)$ -dimensional vector obtained from $\mathbf{z} = (z_1, \dots, z_N) \in \mathbb{Z}_+^N$ by deleting the coordinates i and $i+1$ and by $\mathbf{z}^{(i)}, 1 \leq i \leq N-1$, the $(N-1)$ -dimensional vector obtained from \mathbf{z} by deleting the i -th coordinate. Let $\|\mathbf{x}\|$ be the sum of absolute values of all coordinates of the vector \mathbf{x} .

Set $\mathcal{C}_i = \{\mathbf{z} \in \mathbb{Z}_+^N : \|\mathbf{z}^{(i)}\| > 0\}$, $\mathcal{B}_i = \mathbb{Z}_+^N \setminus \mathcal{C}_i$ and

$$\mathcal{C}_{i,i+1} = \left\{ \mathbf{z} \in \mathbb{Z}_+^N : \|\mathbf{z}^{(i,i+1)}\| > 0 \right\}.$$

Put $Z_i(m) = Z_1(m) + \dots + Z_i(m)$ and denote

$$Z_i(m, n) = \sum_{k=1}^i Z_k(m, n), \quad \bar{Z}_i(m, n) = \sum_{k=i}^N Z_k(m, n).$$

In what follows it will be convenient to write $\mathbf{P}_n(\mathcal{B})$ for $\mathbf{P}(\mathcal{B} | \mathbf{Z}(n) \neq \mathbf{0}, \mathbf{Z}(0) = \mathbf{e}_1)$ for any admissible event \mathcal{B} .

We start checking the desired tightness of the prelimiting processes in Theorems 1 and 2 by proving two important lemmas.

Let $A_i(n) = \{m : n^{\gamma_i} g_n(\gamma_i) \leq m < n^{\gamma_{i+1}-\varepsilon} g_n(\gamma_{i+1} - \varepsilon)\}$, $\varepsilon > 0$.

Lemma 18 *For any $i = 0, 1, 2, \dots, N-1$ and $\varepsilon \in (0, \gamma_1)$*

$$\lim_{n \rightarrow \infty} \mathbf{P}_n(\exists m \in A_i(n) : \mathbf{Z}(m, n) \in \mathcal{C}_{i+1}) = 0.$$

Proof. If $m \in A_i(n)$ then $\bar{Z}_{i+2}(m, n) \leq \bar{Z}_{i+2}(n^{\gamma_{i+1}-\varepsilon} g_n(\gamma_{i+1} - \varepsilon), n)$ and

$$\{\mathbf{Z}_i(m, n) > 0\} \Rightarrow \{\mathbf{Z}_i(m) > 0\} \Rightarrow \{\mathbf{Z}_i(n^{\gamma_i} g_n(\gamma_i)) > 0\}.$$

Thus,

$$\begin{aligned} \mathbf{P}_n(\exists m \in A_i(n) : \mathbf{Z}(m, n) \in \mathcal{C}_{i+1}) &\leq \mathbf{P}_n(\bar{Z}_{i+2}(n^{\gamma_{i+1}-\varepsilon} g_n(\gamma_{i+1} - \varepsilon), n) > 0) \\ &\quad + \mathbf{P}_n(\mathbf{Z}_i(n^{\gamma_i} g_n(\gamma_i)) > 0). \end{aligned}$$

Letting n tend to infinity we see that the first summand at the right-hand side of the inequality vanishes by (36), while the second one is zero for $i = 0$ and tends to zero for $1 \leq i \leq N - 1$ in view of

$$\begin{aligned} \mathbf{P}_n(\mathbf{Z}_{i-1}(n^{\gamma_i} g_n(\gamma_i)) > 0) &= \frac{\mathbf{P}(T_i > n^{\gamma_i} g_n(\gamma_i))}{\mathbf{P}(T_N > n)} \\ &\sim \frac{c_{1i}}{c_{1N}} \frac{n^{1/2^{N-1}}}{(n^{1/2^{N-i}} g_n(\gamma_i))^{1/2^{i-1}}} = \frac{c_{1i}}{c_{1N}} \frac{1}{(g_n(\gamma_i))^{1/2^{i-1}}}. \end{aligned}$$

The lemma is proved.

Lemma 19 *If $N \geq 3$ then for any $i = 1, 2, \dots, N - 1$*

$$\lim_{n \rightarrow \infty} \mathbf{P}_n(\exists m \in [n^{3\gamma_{i-1}}, n^{3\gamma_i}] : \mathbf{Z}(m, n) \in \mathcal{C}_{i,i+1}) = 0.$$

Proof. By the same arguments as in Lemma 18, we conclude

$$\begin{aligned} \mathbf{P}_n(\exists m \in [n^{3\gamma_{i-1}}, n^{3\gamma_i}] : \mathbf{Z}(m, n) \in \mathcal{C}_{i,i+1}) \\ \leq \mathbf{P}_n(\bar{Z}_{i+2}(n^{3\gamma_i}, n) > 0) + \mathbf{P}_n(\mathbf{Z}_{i-1}(n^{3\gamma_{i-1}}) > 0). \end{aligned}$$

According to point 3) of Theorem 13 the first summand tends to zero as $n \rightarrow \infty$ while the second is, by definition zero for $i = 1$ and is evaluated as

$$\frac{\mathbf{P}(T_{i-1} > n^{3\gamma_{i-1}})}{\mathbf{P}(T_N > n)} \sim \frac{c_{1,i-1}}{c_{1N}} \frac{n^{1/2^{N-1}}}{(n^{3/2^{N-i+1}})^{1/2^{i-2}}} \sim \frac{c_{1,i-1}}{c_{1N}} \frac{1}{n^{1/2^{N-2}}}$$

for $i \geq 2$. This completes the proof of the lemma.

6.1 Macroscopic view

In this section we prove Theorem 1 which describes the macroscopic structure of the family tree. Convergence of the finite-dimensional distributions of $\{\mathbf{Z}(n^t g_n(t), n), 0 \leq t < 1\}$ to the respective finite-dimensional distributions of $\{\mathbf{R}(t), 0 \leq t < 1\}$ has been established in (42). Thus, we concentrate on proving the tightness.

Since $\mathbf{Z}(n^t g_n(t), n)$ has integer-valued components we need to check for each interval $A_i = [\gamma_i, \gamma_{i+1} - \varepsilon], i = 0, 1, \dots, N - 1$, that (see [2], Theorem 15.3)

1) for any positive η there exists L such that

$$\mathbf{P}_n \left(\sup_{t \in A_i} \|\mathbf{Z}(n^t g_n(t), n)\| > L \right) \leq \eta, \quad n \geq 1; \quad (49)$$

2) for any positive η there exist $\delta > 0$ and n_0 such that, for all $n \geq n_0$

$$\mathbf{P}_n \left(\max \left(\min_{k=1,2} \|\mathbf{Z}(n^t g_n(t), n) - \mathbf{Z}(n^{t_k} g_n(t_k), n)\| \right) \neq 0 \right) \leq \eta, \quad (50)$$

where the max is taken over all $\gamma_i \leq t_1 \leq t \leq t_2 \leq \gamma_{i+1} - \varepsilon$ such that $t_2 - t_1 \leq \delta$;

$$\mathbf{P}_n(\exists t, s \in [\gamma_i, \gamma_i + \delta] : \mathbf{Z}(n^t g_n(t), n) \neq \mathbf{Z}(n^s g_n(s), n)) \leq \eta, \quad (51)$$

and

$$\mathbf{P}_n(\exists t, s \in [\gamma_{i+1} - \delta - \varepsilon, \gamma_{i+1} - \varepsilon] : \mathbf{Z}(n^t g_n(t), n) \neq \mathbf{Z}(n^s g_n(s), n)) \leq \eta. \quad (52)$$

The fact that the random variable $\|\mathbf{Z}(n^t g_n(t), n)\|$ is monotone in t for fixed n essentially simplifies the proof.

Indeed, in this case

$$\mathbf{P}_n \left(\sup_{t \in A_i} \|\mathbf{Z}(n^t g_n(t), n)\| > L \right) \leq \mathbf{P}_n(\|\mathbf{Z}(n^{1-\varepsilon} g_n(1-\varepsilon), n)\| > L)$$

and (49) follows from the one-dimensional convergence established in (36) for $i = N - 1$.

To prove (50)-(52) we introduce the events

$$\begin{aligned} \mathcal{D}_i &= \{ \forall t \in A_i : \mathbf{Z}(n^t g_n(t), n) \in \mathcal{B}_{i+1} \}, \\ \mathcal{F}_i(a, b) &= \{ \exists t, s \in [a, b] : Z_{i+1}(n^t g_n(t), n) \neq Z_{i+1}(n^s g_n(s), n) \}, \end{aligned}$$

take a sufficiently small $\delta > 0$ and observe that if $[a, b] \subset [\gamma_i, \gamma_{i+1} - \varepsilon]$ then

$$\begin{aligned} \mathbf{P}_n(\exists t, s \in [a, b] : \mathbf{Z}(n^t g_n(t), n) \neq \mathbf{Z}(n^s g_n(s), n)) \\ \leq \mathbf{P}_n(\exists t \in A_i : \mathbf{Z}(n^t g_n(t), n) \in \mathcal{C}_{i+1}) + \mathbf{P}_n(\mathcal{D}_i \cap \mathcal{F}_i(\gamma_i, \gamma_{i+1} - \varepsilon)). \end{aligned}$$

By Lemma 18 the first term at the right-hand side tends to zero as $n \rightarrow \infty$.

Further, for $i \geq 1$

$$\mathbf{P}_n(\mathcal{D}_i \cap \mathcal{F}_i(\gamma_i, \gamma_{i+1} - \varepsilon)) \leq \mathbf{P}_n(Z_{i+1}(n^{\gamma_i} g_n, n) \neq Z_{i+1}(n^{\gamma_{i+1} - \varepsilon} g_n, n)) \rightarrow 0$$

by (42). This justifies (51)-(52).

To check the validity of (50) it remains to note that

$$\begin{aligned} \mathbf{P}_n \left(\max \left(\min_{k=1,2} \|\mathbf{Z}(n^t g_n(t), n) - \mathbf{Z}(n^{t_k} g_n(t_k), n)\| \right) \neq 0 \right) \\ \leq \mathbf{P}_n(\exists t, s \in [\gamma_i, \gamma_{i+1} - \varepsilon] : \mathbf{Z}(n^t g_n(t), n) \neq \mathbf{Z}(n^s g_n(s), n)) \end{aligned}$$

and to use the same arguments as before.

Theorem 1 is proved.

6.2 Microscopic view

We follow in this section the ideas of paper [6] and to this aim formulate a particular and slightly modified case of Theorem 6.5.4 in [4] giving a convergence criterion in Skorokhod topology for a class of Markov processes.

Let $\mathbf{K}_n(y), n = 1, 2, \dots$ be a sequence of Markov processes with values in \mathbb{Z}_+^N whose trajectories belong with probability 1 to the space $D_{[a,b]}(\mathbb{Z}_+^N)$ of cadlag functions on $[a, b]$.

Theorem 20 *If the finite-dimensional distributions of $\{\mathbf{K}_n(y), a \leq y \leq b\}$ converge, as $n \rightarrow \infty$, to the respective finite-dimensional distributions of a process $\{\mathbf{K}(y), a \leq y \leq b\}$ and there exists a partition $\mathbb{Z}_+^N = \mathcal{B} \cup \mathcal{C}, \mathcal{B} \cap \mathcal{C} = \emptyset$ such that*

$$\lim_{h \downarrow 0} \overline{\lim_{n \rightarrow \infty}} \sup_{0 \leq s-y \leq h} \sup_{\mathbf{z} \in \mathcal{B}} \mathbf{P}(\mathbf{K}_n(s) \neq \mathbf{K}_n(y) | \mathbf{K}_n(y) = \mathbf{z}) = 0,$$

and

$$\lim_{n \rightarrow \infty} \mathbf{P}(\exists y \in [a, b] : \mathbf{K}_n(y) \in \mathcal{C}) = 0$$

then, as $n \rightarrow \infty$

$$\mathcal{L}\{\mathbf{K}_n(y), a \leq y \leq b\} \Longrightarrow \mathcal{L}\{\mathbf{K}(y), a \leq y \leq b\}.$$

In view of Lemma 16 the law $\mathbf{P}_n(\{\mathbf{Z}(m, n), 0 \leq m \leq n\} \in (\cdot) | \mathbf{Z}(n) \neq \mathbf{0})$ specifies, for each fixed n an inhomogeneous Markov branching process. We denote its transition probabilities by $\mathbf{P}_n(m_1, \mathbf{z}; m_2, (\cdot))$.

Proving the tightness of $\mathbf{U}_i(\cdot), i = 1, 2, \dots, N$, we need to construct an appropriate partition of \mathbb{Z}_+^N and to use Theorem 20 for each $[0, b] \subset [0, \infty)$.

Observe that if $\mathbf{w} = (w_1, \dots, w_N) \leq \mathbf{z} = (z_1, \dots, z_N)$ (where the inequality is understood componentwise) then

$$\mathbf{P}_n(m_0, \mathbf{w}; m_1, \{\mathbf{w}\}) \geq \mathbf{P}_n(m_0, \mathbf{z}; m_1, \{\mathbf{z}\}).$$

Let $\mathcal{C}(k) = \{\mathbf{z} \in \mathbb{Z}_+^N : \|\mathbf{z}\| \leq k\},$

$$\mathcal{C}_i(k) = \{\mathbf{z} \in \mathbb{Z}_+^N : z_1 + \dots + z_{i-1} > 0; \|\mathbf{z}\| \leq k\}, \mathcal{J}_i(k) = \mathcal{C}(k) \setminus \mathcal{C}_i(k).$$

Fix $i \in \{1, \dots, N-1\}$ and denote $m_j = (Y_j + l_n)n^{\gamma_i}, j = 1, 2$.

Lemma 21 *Under Hypothesis A for any fixed k and $0 < b < \infty$*

$$\lim_{h \downarrow 0} \overline{\lim_{n \rightarrow \infty}} \sup_{\substack{0 \leq Y_1 - Y_0 \leq h, \\ Y_1, Y_0 \in [0, b]}} \sup_{\mathbf{z} \in \mathcal{J}_i(k)} \mathbf{P}_n(\mathbf{Z}(m_1; n) \neq \mathbf{z} | \mathbf{Z}(m_0; n) = \mathbf{z}) = 0.$$

Proof. By the branching property, the decomposability assumption, and the positivity of the offspring number of each particle in the reduced process we have for all $m_1 \geq m_0$ and $\mathbf{z} \in \mathcal{J}_i(k)$

$$\begin{aligned} \mathbf{P}_n(m_0, \mathbf{z}; m_1, \{\mathbf{z}\}) &= \prod_{j=i}^N (\mathbf{P}_n(m_0, \mathbf{e}_j; m_1, \{\mathbf{e}_j\}))^{z_j} \\ &\geq \prod_{j=i}^N (\mathbf{P}_n(m_0, \mathbf{e}_j; m_1, \{\mathbf{e}_j\}))^k. \end{aligned}$$

Using Lemma 17 we get for $m_0 = (Y_0 + l_n)n^{\gamma_i}$ and $m_1 = (Y_1 + l_n)n^{\gamma_i}$:

$$\inf_{\substack{0 \leq Y_1 - Y_0 \leq h, \\ Y_0, Y_1 \in [0, b]}} \mathbf{P}_n(m_0, \mathbf{z}; m_1, \{\mathbf{z}\}) \geq (1 - \chi h)^{Nk}. \quad (53)$$

This implies the claim of the lemma.

Lemma 22 *If $m_j = (Y_j + l_n)n^{\gamma_i}$, $j = 0, 1, 2$, and $0 \leq Y_0 < Y_1 < Y_2$ with $Y_1 - Y_0 \leq h$, then for all $n \geq n_0$*

$$\begin{aligned} & \mathbf{P}_n(\mathbf{Z}(m_1, n) = \mathbf{z} | \mathbf{Z}(m_0, n) = \mathbf{z}; \|\mathbf{Z}(m_2, n)\| \leq k) \\ & \geq \mathbf{P}_n(m_0, \mathbf{z}; m_1, \{\mathbf{z}\}) \frac{\mathbf{P}_n(m_1, \mathbf{z}; m_2, \mathcal{C}(k))}{\mathbf{P}_n(m_1, \mathbf{z}; m_2, \mathcal{C}(k)) + \chi Nkh}. \end{aligned}$$

Proof. We have

$$\begin{aligned} & \mathbf{P}_n(\mathbf{Z}(m_1, n) = \mathbf{z} | \mathbf{Z}(m_0, n) = \mathbf{z}, \|\mathbf{Z}(m_2, n)\| \leq k) \\ & = \mathbf{P}_n(m_0, \mathbf{z}; m_1, \{\mathbf{z}\}) \frac{\mathbf{P}_n(m_1, \mathbf{z}; m_2, \mathcal{C}(k))}{\mathbf{P}_n(m_0, \mathbf{z}; m_2, \mathcal{C}(k))}. \end{aligned}$$

In view of (53)

$$\begin{aligned} \mathbf{P}_n(m_0, \mathbf{z}; m_2, \mathcal{C}(k)) &= \sum_{\mathbf{z}'} \mathbf{P}_n(m_0, \mathbf{z}; m_1, \{\mathbf{z}'\}) \mathbf{P}_n(m_1, \mathbf{z}'; m_2, \mathcal{C}(k)) \\ &\leq 1 - \mathbf{P}_n(m_0, \mathbf{z}; m_1, \{\mathbf{z}\}) + \mathbf{P}_n(m_1, \mathbf{z}; m_2, \mathcal{C}(k)) \\ &\leq 1 - (1 - \chi h)^{Nk} + \mathbf{P}_n(m_1, \mathbf{z}; m_2, \mathcal{C}(k)) \\ &\leq \chi Nkh + \mathbf{P}_n(m_1, \mathbf{z}; m_2, \mathcal{C}(k)). \end{aligned}$$

Hence the needed statement follows.

Lemma 23 *Under Hypothesis A for any fixed k , $0 < b < \infty$, and $m_0 = (Y_0 + l_n)n^{\gamma_i}$, $m_1 = (Y_1 + l_n)n^{\gamma_i}$, $m_2 = 2bn^{\gamma_i}$ we have*

$$\lim_{h \downarrow 0} \overline{\lim_{n \rightarrow \infty}} \sup_{\substack{0 \leq Y_1 - Y_0 \leq h \\ Y_1, Y_0 \in [0, b]}} \sup_{\mathbf{z} \in \mathcal{I}_i(k)} \mathbf{P}_n(\mathbf{Z}(m_1, n) \neq \mathbf{z} | \mathbf{Z}(m_0, n) = \mathbf{z}, \|\mathbf{Z}(m_2, n)\| \leq k) = 0.$$

Proof. By (53) and Lemma 22 for $m_0 = (Y_0 + l_n)n^{\gamma_i}$ and $m_1 = (Y_1 + l_n)n^{\gamma_i}$

$$\begin{aligned} & \mathbf{P}_n(\mathbf{Z}(m_1, n) = \mathbf{z} | \mathbf{Z}(m_0, n) = \mathbf{z}; \|\mathbf{Z}(m_2, n)\| \leq k) \\ & \geq (1 - \chi h)^{Nk} \frac{\mathbf{P}_n(m_1, \mathbf{z}; m_2, \mathcal{C}(k))}{\mathbf{P}_n(m_1, \mathbf{z}; m_2, \mathcal{C}(k)) + \chi Nkh}. \end{aligned}$$

Using the decomposability hypothesis and Lemma 17 we obtain

$$\begin{aligned} & \mathbf{P}_n(m_1, \mathbf{z}; m_2, \mathcal{C}(k)) \geq \mathbf{P}_n(m_1, \mathbf{z}; m_2, \{\mathbf{z}\}) \\ & = \prod_{j=i}^N (\mathbf{P}_n(m_1, \mathbf{e}_j; m_2, \{\mathbf{e}_j\}))^{z_j} \geq \prod_{j=i}^N \mathbf{P}_n^k(l_n n^{\gamma_i}, \mathbf{e}_j; 2bn^{\gamma_i}, \{\mathbf{e}_j\}). \end{aligned}$$

It follows from Theorem 14 that

$$\lim_{n \rightarrow \infty} \prod_{j=i}^N \mathbf{P}_n^k(l_n n^{\gamma_i}, \mathbf{e}_j; 2bn^{\gamma_i}, \{\mathbf{e}_j\}) = \mathbf{P}^k(\mathbf{U}_i(2b) = \mathbf{e}_i | \mathbf{U}_i(0) = \mathbf{e}_i) = B > 0.$$

Hence we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} \inf_{\substack{0 \leq Y_1 - Y_0 \leq h \\ Y_1, Y_0 \in [0, b]}} \inf_{\mathbf{z} \in \mathcal{J}_i(k)} \mathbf{P}_n(\mathbf{Z}(m_1; n) = \mathbf{z} | \mathbf{Z}(m_0; n) = \mathbf{z}, \|\mathbf{Z}(m_2, n)\| \leq k) \\ & \geq (1 - \chi h)^{Nk} \frac{B}{B + \chi N k h}. \end{aligned}$$

Letting $h \downarrow 0$ completes the proof of the lemma.

Corollary 24 *Under the conditions of Lemma 23*

$$\begin{aligned} & \mathcal{L} \left\{ \mathbf{Z}((y + l_n)n^{1/2^{N-i}}, n), 0 \leq y \leq b \mid \|\mathbf{Z}(m_2, n)\| \leq k, \mathbf{Z}(n) \neq \mathbf{0} \right\} \\ & \implies \mathcal{L}_{R_i} \{ \mathbf{U}_i(y), 0 \leq y \leq b \mid \|\mathbf{U}_i(2b)\| \leq k \}. \end{aligned}$$

Proof. Convergence of finite-dimensional distributions follows from the respective results for the convergence of the processes established in point 1) of Theorem 2. Tightness follows from Lemma 23 and Theorem 20 by taking $\mathcal{B} = \mathcal{J}_i(k)$ and $\mathcal{C} = \mathcal{C}_i(k)$ and observing that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbf{P}_n(\mathbf{Z}(l_n n^{\gamma_i}, n) \in \mathcal{C}_i(k) \mid \|\mathbf{Z}(m_2, n)\| \leq k) \\ & \leq \lim_{n \rightarrow \infty} \mathbf{P}_n(\mathbb{Z}_{i-1}(l_n n^{\gamma_i}) > 0 \mid \|\mathbf{Z}(m_2, n)\| \leq k) = 0. \end{aligned}$$

Proof of Theorem 2. Let for $c > b$

$$\begin{aligned} \mathbf{P}_{n,i}(b; (\cdot)) &= \mathbf{P}_n(\{\mathbf{Z}((y + l_n)n^{\gamma_i}, n), 0 \leq y \leq b\} \in (\cdot)), \\ \mathbf{P}_{n,i}^{(k)}(b, c; (\cdot)) &= \mathbf{P}_n(\{\mathbf{Z}((y + l_n)n^{\gamma_i}, n), 0 \leq y \leq b\} \in (\cdot) \mid \|\mathbf{Z}(cn^{\gamma_i}, n)\| \leq k), \\ \bar{\mathbf{P}}_{n,i}^{(k)}(b, c; (\cdot)) &= \mathbf{P}_n(\{\mathbf{Z}((y + l_n)n^{\gamma_i}, n), 0 \leq y \leq b\} \in (\cdot) \mid \|\mathbf{Z}(cn^{\gamma_i}, n)\| > k) \end{aligned}$$

and

$$\begin{aligned} \mathcal{P}_i(b; (\cdot)) &= \mathbf{P}_{R_i}(\{\mathbf{U}_i(y), 0 \leq y \leq b\} \in (\cdot)), \\ \mathcal{P}_i^{(k)}(b, c; (\cdot)) &= \mathbf{P}_{R_i}(\{\mathbf{U}_i(y), 0 \leq y \leq b\} \in (\cdot) \mid \|\mathbf{U}_i(c)\| \leq k). \end{aligned}$$

Then for $0 < b < \infty$ and a continuous real function ψ on $D_{[0,b]}(\mathbb{Z}_+^N)$ such that $|\psi| \leq q$ for a positive q we have

$$\begin{aligned} \int \psi(x) \mathbf{P}_{n,i}(b; dx) &= \mathbf{P}_n(\|\mathbf{Z}(2bn^{\gamma_i}, n)\| > k) \int \psi(x) \bar{\mathbf{P}}_{n,i}^{(k)}(b, 2b; dx) \\ &\quad + \mathbf{P}_n(\|\mathbf{Z}(2bn^{\gamma_i}, n)\| \leq k) \int \psi(x) \mathbf{P}_{n,i}^{(k)}(b, 2b; dx). \end{aligned}$$

For the first summand we get

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \mathbf{P}_n(\|\mathbf{Z}(2bn^{\gamma_i}, n)\| > k) \int \psi(x) \bar{\mathbf{P}}_{n,i}^{(k)}(b, 2b; dx) \\ & \leq q \limsup_{n \rightarrow \infty} \mathbf{P}_n(\|\mathbf{Z}(2bn^{\gamma_i}, n)\| > k) = q \mathbf{P}_{R_i}(\|\mathbf{U}_i(2b)\| > k) = o(1) \end{aligned}$$

as $k \rightarrow \infty$ by the properties of $\mathbf{U}_i(\cdot)$.

On the other hand, letting first $n \rightarrow \infty$ and then $k \rightarrow \infty$ we obtain

$$\begin{aligned} & \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbf{P}_n(\|\mathbf{Z}(2bn^{\gamma_i}, n)\| \leq k) \int \psi(x) \mathbf{P}_{n,i}^{(k)}(b, 2b; dx) \\ & = \lim_{k \rightarrow \infty} \mathbf{P}_{R_i}(0 < \|\mathbf{U}_i(2b)\| \leq k) \int \psi(x) \mathcal{P}_i^{(k)}(b, 2b; dx) \\ & = \lim_{k \rightarrow \infty} \int_{\{0 < \|\mathbf{U}_i(2b)\| \leq k\}} \psi(x) \mathcal{P}_i(b, 2b; dx) = \int \psi(x) \mathcal{P}_i(b; dx). \end{aligned}$$

Thus,

$$\lim_{n \rightarrow \infty} \int \psi(x) \mathbf{P}_n(\mathbf{Z}((\cdot + l_n)n^{\gamma_i}, n) \in dx) = \int \psi(x) \mathcal{P}_i(b; dx)$$

for any bounded continuous function on $D_{[0,b]}(\mathbb{Z}_+^N)$ proving point 1) of Theorem 2.

The proof of point 2) of Theorem 2 needs only a few changes in comparison with the proof of the respective theorem in [6] and we omit it.

7 Proofs of Theorems 3 and 4

Proof of Theorem 3. Our arguments are based on the following simple observation

$$\{\bar{Z}_1(m, n) = 1\} \Leftrightarrow \{\beta_n \geq m\}.$$

Proof of 1). According to (35) for $m \ll n^{\gamma_1}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{P}_n(\bar{Z}_1(m, n) = 1) &= \lim_{n \rightarrow \infty} \mathbf{P}_n(Z_1(m, n) = 1) \\ &+ \lim_{n \rightarrow \infty} \mathbf{P}_n(\bar{Z}_2(m, n) = 1) = 1 + 0 = 1. \end{aligned}$$

Proof of 2). Observe that by point 2) of Theorem 13

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{P}_n(\beta_n \geq yn^{\gamma_i}) &= \lim_{n \rightarrow \infty} \mathbf{P}_n(\bar{Z}_1(yn^{\gamma_i}, n) = 1) \\ &= \lim_{n \rightarrow \infty} \mathbf{P}_n(Z_i(yn^{\gamma_i}, n) + Z_{i+1}(yn^{\gamma_i}, n) = 1) \\ &= \lim_{n \rightarrow \infty} \mathbf{P}_n(Z_i(yn^{\gamma_i}, n) = 1) + \lim_{n \rightarrow \infty} \mathbf{P}_n(Z_{i+1}(yn^{\gamma_i}, n) = 1). \end{aligned}$$

Direct calculations show that

$$-\frac{\partial \varphi_i(y; s_i, s_{i+1})}{\partial s_i} \Big|_{s_i=s_{i+1}=0} = \frac{1 - \tanh(yb_i c_{iN})}{1 + \tanh(yb_i c_{iN})} = e^{-2yb_i c_{iN}}$$

and

$$-\frac{\partial \varphi_i(y; s_i, s_{i+1})}{\partial s_{i+1}} \Big|_{s_i=s_{i+1}=0} = \frac{\tanh(yb_i c_{iN})}{1 + \tanh(yb_i c_{iN})} = \frac{1}{2} - \frac{1}{2} e^{-2yb_i c_{iN}}.$$

Thus,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{P}_n(Z_i(yn^{\gamma_i}, n) = 1; \beta_n \geq yn^{\gamma_i}) \\ = -\frac{\partial(\varphi_i(y; s_i, s_{i+1}))^{1/2^{i-1}}}{\partial s_i} \Big|_{s_i=s_{i+1}=0} = \frac{1}{2^{i-1}} e^{-2yb_i c_{iN}} \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{P}_n(Z_{i+1}(yn^{\gamma_i}, n) = 1; \beta_n \geq yn^{\gamma_i}) \\ = -\frac{\partial(\varphi_i(y; s_i, s_{i+1}))^{1/2^{i-1}}}{\partial s_{i+1}} \Big|_{s_i=s_{i+1}=0} = \frac{1}{2^i} (1 - e^{-2yb_i c_{iN}}). \end{aligned}$$

Combining the previous estimates yields

$$\lim_{n \rightarrow \infty} \mathbf{P}_n(\beta_n \leq yn^{\gamma_i}) = 1 - \frac{1}{2^i} - \frac{1}{2^i} e^{-2yb_i c_{iN}}.$$

Proof of 3). This is evident.

Proof of 4). The needed statement follows from the equality

$$-\frac{\partial}{\partial s_N} \left(\frac{1}{x + (1-x)/(1-s_N)} \right)^{2^{-(N-1)}} \Big|_{s_N=0} = \frac{1}{2^{N-1}} (1-x).$$

Proof of Theorem 4. Consider the case $N \geq 4$ and $i \in \{2, 3, \dots, N-2\}$ only. For $N = 2, 3$ or $N \geq 4$ and $i \in \{1, N-1\}$ some of the random variables (events) below do not exist (are empty) and the needed arguments become shorter.

Since the total number of particles of all types in the reduced process does not decrease with time, $\mathbf{P}_n(\beta_n < m) = \mathbf{P}_n(\bar{Z}_1(m, n) \geq 2)$. We now take

$$m_i = n^{\gamma_i(1+\gamma_i)}, \quad i = 1, 2, \dots, N-1,$$

and denote $\mathcal{H}_i = \{m : m_{i-1} \leq m \leq m_i\}$.

Since $\bar{Z}_i(k, n)$ is monotone increasing in k for each fixed n , Theorem 13 and (15) imply, as $n \rightarrow \infty$

$$\begin{aligned} \mathbf{P}_n(\zeta_n = i; \beta_n \notin \mathcal{H}_i) &\leq \mathbf{P}_n(\exists k < m_{i-1} : Z_i(k, n) > 0) \\ &\quad + \mathbf{P}_n(\exists k > m_i : Z_i(k, n) > 0) \\ &\leq \mathbf{P}_n(\bar{Z}_i(m_{i-1}, n) > 0) + \mathbf{P}_n(Z_i(m_i) > 0) = o(1). \end{aligned}$$

By the same statements we conclude, as $n \rightarrow \infty$

$$\begin{aligned} \mathbf{P}_n(\zeta_n \notin \{i, i+1\}; \beta_n \in \mathcal{H}_i) &\leq \mathbf{P}_n(\exists k \in \mathcal{H}_i : Z_{i-1}(k, n) + \bar{Z}_{i+2}(k, n) > 0) \\ &\leq \mathbf{P}_n(\exists k \in \mathcal{H}_i : Z_{i-1}(k) + \bar{Z}_{i+2}(k, n) > 0) \\ &\leq \mathbf{P}_n(\exists k \in \mathcal{H}_i : Z_{i-1}(k) > 0) + \mathbf{P}_n(\exists k \in \mathcal{H}_i : \bar{Z}_{i+2}(k, n) > 0) \\ &\leq \mathbf{P}_n(\bar{Z}_{i-1}(m_{i-1}) > 0) + \mathbf{P}_n(\bar{Z}_{i+2}(m_i, n) > 0) = o(1). \end{aligned}$$

Hence, as $n \rightarrow \infty$

$$\begin{aligned}\mathbf{P}_n(\zeta_n = i) &= \mathbf{P}_n(\zeta_n = i; \beta_n \in \mathcal{H}_i) + o(1) \\ &= \mathbf{P}_n(\beta_n \in \mathcal{H}_i) - \mathbf{P}_n(\zeta_n = i+1; \beta_n \in \mathcal{H}_i) + o(1).\end{aligned}\quad (54)$$

Introduce the event

$$\mathcal{G}_i(j, n) = \{Z_i(j, n) + \bar{Z}_{i+2}(j+1, n) = 0; Z_{i+1}(j, n) = 1\}.$$

Clearly,

$$\begin{aligned}\mathbf{P}_n(\zeta_n = i+1; \beta_n \in \mathcal{H}_i) &= \sum_{j=m_{i-1}}^{m_i} \mathbf{P}_n(\zeta_n = i+1; \beta_n = j) \\ &= \sum_{j=m_{i-1}}^{m_i} \mathbf{P}_n(\mathcal{G}_i(j, n), \bar{Z}_{i+1}(j+1, n) \geq 2) \\ &= o(1) + \sum_{j=m_{i-1}}^{m_i} \mathbf{P}_n(\mathcal{G}_i(j, n)) \mathbf{P}_n(Z_{i+1}(j+1, n) \geq 2 | \mathbf{Z}(j, n) = \mathbf{e}_{i+1}).\end{aligned}$$

It is not difficult to check (recall (2), (44) and (45)) that

$$\begin{aligned}\mathbf{P}_n(Z_{i+1}(j+1, n) = 1 | \mathbf{Z}(j, n) = \mathbf{e}_{i+1}) &= \frac{Q_{n-j-1}^{(i+1, N)} \frac{dh_{i+1}(s, \mathbf{1}^{(N-i-1)})}{ds}}{Q_{n-j}^{(i+1, N)}} \Big|_{s=H_{n-j-1}^{(i+1, N)}(\mathbf{0})} \\ &\geq \frac{dh_{i+1}(s, \mathbf{1}^{(N-i-1)})}{ds} \Big|_{s=H_{n-j-1}^{(i+1, N)}(\mathbf{0})} \\ &\geq 1 - 2b_{i+1}Q_{n-j-1}^{(i+1, N)} \\ &\geq 1 - 2b_{i+1}Q_{n-m_i}^{(i+1, N)}.\end{aligned}$$

Hence, using the estimate

$$\begin{aligned}\mathbf{P}_n(Z_{i+1}(j+1, n) \geq 2 | \mathbf{Z}(j, n) = \mathbf{e}_{i+1}) &= 1 - \mathbf{P}_n(Z_{i+1}(j+1, n) = 1 | \mathbf{Z}(j, n) = \mathbf{e}_{i+1}) \\ &\leq 2b_i Q_{n-m_i}^{(i+1, N)}\end{aligned}$$

we conclude

$$\begin{aligned}\mathbf{P}_n(\zeta_n = i+1; \beta_n \in \mathcal{H}_i) &= o(1) + O(m_i Q_{n-m_i}^{(i+1, N)}) \\ &= o(1) + O(n^{\gamma_i(1+\gamma_i)} n^{-\gamma_{i+1}}) = o(1).\end{aligned}$$

This, on account of (11) and (54) gives

$$\lim_{n \rightarrow \infty} \mathbf{P}_n(\zeta_n = i) = \lim_{n \rightarrow \infty} \mathbf{P}_n(\beta_n \in \mathcal{H}_i) = \lim_{n \rightarrow \infty} \mathbf{P}_n(n^{\gamma_i} \ll \beta_n \ll n^{\gamma_{i+1}}) = \frac{1}{2^i}$$

as desired.

Finally,

$$\lim_{n \rightarrow \infty} \mathbf{P}_n(\zeta_n = N) = 1 - \sum_{i=1}^{N-1} \frac{1}{2^i} = \frac{1}{2^{N-1}}.$$

Theorem 4 is proved.

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